A Simple Theory of Why and When Firms Go Public

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Abstract

We introduce a simple model of a firm's optimal investment, dividend, and debt and equity financing decisions to address the key questions of why and when private firms choose to "go public" via an initial public offering (IPO) of their shares on a public stock exchange. We characterize the optimal policy of a privately held firm and show how an owner's desire to consumption smooth distorts the firm's investment and dividend policy, resulting in a loss of market value relative to a publicly owned firm with a comparable level of capital and debt. We introduce a new fixed point characterization of the level of new equity raised in a seasoned equity offering (SEO) or an IPO. We answer the question of "why go public" by characterizing the conditions under which the owner of a private firm will want to undertake an IPO relative to other options (such as borrowing from a bank or investment of retained earnings, while keeping the firm private). When the owner decides to undertake an IPO, we characterize how much of the IPO proceeds the original owner cashes out, how much will be reinvested in the firm, and how large of an ownership stake the owner retains in the newly formed public firm. We address the question of when to go public by showing that only young firms of moderate size (not too small and not too big) have the highest gains from going public.

Keywords: IPOs, investment, finance, capital structure, theory of the firm, dynamic programming

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1 Introduction

We introduce a simple dynamic model of the firm to provide a new theory of why and when the owner of a private firm chooses to take their firm public by holding an initial public offering (IPO) where the company's shares are sold on a stock exchange. The model we consider is simple enough to provide an analytical solution and full characterization of the optimal investment and dividend policy of a public firm which invests in a single illiquid capital good k. We use the term "public firm" to distinguish it from a "private firm" which we also analyze. The key difference is that a public firm's objective is to adopt an investment and borrowing policy to maximize a discounted stream of *dividends*, whereas a private firm adopts an investment and financial policy to maximize a discounted stream of *utilities*.

A standard explanation of why firms go public is the need for capital to finance their growth. While our model incorporates this key motive, we provide a different explanation of why private firms go public that can be framed as an application of the classic *separation theorem* of finance. We show that the owner of a private firm adopts an inefficient investment and dividend policy, owing to their desire to *consumption smooth* which in turn leads to a corresponding incentive to *dividend smooth*. The separation theorem tells us that if the private owner does not get a direct utility from controlling the firm, and if financial markets are sufficiently complete, it is better to take his firm public and use financial markets to smooth dividends.

However a successful theory must also explain why some private firms choose *not* to go public, and the *agency problems* associated with a separation of ownership and management of public companies is a key explanation. The seminal paper by Jensen and Meckling [1976] emphasizes "why an entrepreneur or manager in a firm which has a mixed financial structure (containing both debt and equity claims) will choose a set of activities for the firm such that the total value of the firm is *less* than it would be if he were the sole owner and why this result is independent of whether the firm operates in a monopolistic or competitive product or factor markets."

Our theory of IPOs stands the traditional agency theoretic explanation on its head. We abstract from the agency problems of public ownership that Jensen and Meckling [1976] focused on since we agree with Fama and Jensen [1983] that the separation of decision and risk-bearing functions observed in large corporations "survives in these organizations in part because of the benefits of specialization of management and risk bearing but also because of an effective common approach to controlling the agency problems caused by separation and risk-bearing functions." (p. 301-302). Thus, we follow Fama and Jensen [1983] by assuming that to a first approximation corporations effectively solve their agency problems and operate as discounted dividend maximizers. On the other hand, while private firms do not suffer from the agency problems stemming from a separation of ownership and control, we show that private ownership leads to other constraints and inefficiencies that have not received as much attention in the finance literature.

We frame the IPO decision in the context of a growth model where the owner of a private firm can finance investment via retained earnings and via external debt. Of course, equity financing — via the decision to go public — is a third way that a private owner can finance growth. We show that a private owner can achieve "leverage" via an IPO that is similar in many respects to the leveraging effects of debt finance. In fact, public firms can also finance growth by selling new shares of stock, and we characterize conditions where existing owners of a public firm can be better off by issuing new shares rather than borrowing, which is parallel to our analysis of how a private owner can be better off by taking his firm public via an IPO.

Debt policy is complex but we analyze the model in the presence of "single period debt" where existing debt can be "rolled over" and refinanced with another single period loan. We consider situations where a firm uses both debt and retained earnings to finance its investment, but under our simplifying assumptions, the firms we study will never find it optimal to hold cash balances, but rather will either invest all cash or pay it out as dividends to shareholders.

We show how firms that start with little initial capital have a desire to borrow in order to accelerate their accumulation of capital. Financing investment by debt significantly shortens the time it takes the firm to achieve "optimal scale" compared to a firm that faces liquidity constraints and is unable to borrow. Thus, access to credit markets significantly enhances the growth and value of sufficiently small firms, but has little advantage for larger firms that have accumulated sufficient capital.

Although the earliest work on corporate finance theory date to Fisher [1930] and the work on capital structure by Modigliani and Miller [1958], the theoretical literature on the decision to go public is comparatively small and recent including Pagano [1993], Zingales [1995], Chemmanur and Fulghieri [1999] and Maksimovic and Pichler [2001]. The earliest analyses used two period models to explain both why private firms go public (e.g. as a "result of a value-maximizing decision made by an initial owner who wants to eventually sell his company." Zingales [1995] p. 426), and when to go public, (e.g. "The equilibrium timing of the going-public decision is determined by the firms trade-off between minimizing the duplication in information production by outsiders ... and avoiding the risk-premium demanded by venture capitalists." (Chemmanur and Fulghieri [1999], p. 249). The model by Chemmanur and Fulghieri [1999] follows the agency-theory tradition of Jensen and Meckling [1976] and predicts that "only firms whose entrepreneurs have accumulated a significant track record for successful operation (and thereby a reputation to lose if they engage in value-reducing actions) will find it optimal to sell shares in the public equity market, while those without such a track record will raise capital from private equity investors." (p. 273), whereas Maksimovic and Pichler [2001] relates the IPO decision to technological innovation in a two stage framework where a firm acquires a technology and discovers if it is viable in the first stage and decides how to finance and invest in the firm in the second stage.

In their survey on investment banking and securities issuance, Ritter and Welch [2002] wrote "There are many tradeoffs, but the literature does not have a full model that can explain i) at what stage of a firms life-cycle it is optimal to go public; and ii) why the volume of IPO varies dramatically across time and across countries." However in the years since the Ritter and Welch survey was written we have not seen new and more satisfactory theoretical explanations of *why* and *when* private firms go public that address these issues, especially in the context of multi-period models of *firm growth* where private owners have several options for financing investment, and repeatedly face the discrete dynamic decision whether to take their company public or not.

Our paper builds a dynamic, infinite horizon model of the lifecycle of a firm to explain the tradeoffs that private owners face about whether and when to go public in a rich setting with endogenous investment dilution, dividends, and borrowing choice. However to keep our theory simple and tractable, we abstract from a number of potentially important issues such as agency problems and the role of asymmetric information. We do not explicitly model the role of financial intermediaries such as underwriters and venture capitalists. Instead, we take the fixed and proportional costs of going public as given, without attempting to build a competitive model of underwriting where these costs are determined endogenously, as equilibrium outcomes. We refer readers to Ritter [1987], Ritter and Welch [2002], Draho [2004] and Ljungqvist and Wilhelm [2002] for more detailed descriptions of IPOs and the underwiting process.

We model the IPO decision as an *optimal stopping problem* where the decision to "stop" corresponds to the owner's decision to stop operating as a private firm and instead to take the firm public via an IPO.¹ While a private firm can finance its growth via retained earnings and borrowing, we characterize conditions where raising new capital via an IPO can result in higher payoffs to the owner than borrowing, despite the high fees that investment banks charge to underwrite an IPO, which can be higher than 30% of total IPO proceeds when IPO "underpricing" is taken into account (Ritter [1987]).

The closest antecedents to our paper are the dynamic models of firm investment, financing and growth of Whited [1992] and Cooley and Quadrini [2001], and the dynamic models of IPOs in the unpublished working paper of Clementi [2002] and the published papers of Benninga et al. [2005] and Pastor et al. [2009]. The latter three papers develop dynamic models of firm behavior in which the decision to go public is modeled explicitly. Clementi's model is motivated by empirical evidence that firm operating performance falters in the years after an IPO, and his model "predicts that the operating performance reaches its peak in the period before the offering and experiences a sudden decline at the IPO date." The model of Benninga et al. [2005] and Pastor et al. [2009] focus on the tradeoff between the option value of remaining private and the liquidity and diversification value to the founder from going public. However

¹Technically, the decision to go public is not necessarily an absorbing state, since it is possible for a public firm to *go private* such as Henry Ford did in 1919 via a \$125 million buy-out of minority shareholders after initially taking Ford Motor Company public in 1903, according to *Wikipedia*.

their models abstract from investment and borrowing choices. Our dynamic model also has the trade-off between diversification gains and private benefits of control, but we endogenize investment and borrowing choices via an explicit model of financing and growth similar to Whited [1992] and Cooley and Quadrini [2001]. We introduce a new fixed point condition determining the amount of funds raised in a seasoned equity offering (SEO) or an IPO that we believe is a new contribution to the literature.

Our theory is also motivated by a separate structural empirical analysis of the IPO decision using panel data on Indian firms, Gupta and Rust [2018]. Their dynamic, stochastic model of the IPO decision is quite rich and has the potential to explain several different features and puzzling aspects of IPOs that we see in the data, but the identification of the model is challenging. In particular, there are multiple ways to rationalize why some private firms go public relatively soon after they are founded, while others never do. One explanation for these different outcomes is that some private owners have more optimistic beliefs about their future growth prospects than others. However the costs of IPOs and the regulatory and reporting burdens on public firms are also important factors that can convince even highly successful and profitable private firms to remain private.

In section 2 we review the empirical literature on the decision of why and when to go public to provide a set of "stylized facts" to motivate the formulation of our theoretical model. In section 3, we start out with the simplest context by deriving the optimal investment and financial policy of a public firm that has access to perfect, frictionless financial markets (i.e. no borrowing limits and no transactions costs on debt or equity finance). We characterize the conditions under which a public firm would choose to finance its growth via issuance of more shares versus debt. In section 4 we consider the case of a private firm where we initially rule out the option to go public. We characterize the optimal investment and financial policy of a private owner and show that the owner chooses to *dividend smooth*. We prove a *separation theorem* that shows that the private owner can more effectively smooth his consumption and achieve a higher level of consumption by selling the firm and using annuities to achieve a perfectly flat consumption profile. This provides a first explanation of why private firms go public. In section 5 we formulate the private owner's decision of whether and when to go public. This includes as a subproblem the owner's decision about what ownership share to retain in the post-IPO firm and what share of the IPO proceeds to re-invest in the newly formed public firm. In section 6 we consider various extensions of the basic model to allow for a) non-concave production functions, b) stochastic production shocks and Bayesian learning about the productivity of the firm, and c) single period debt with borrowing constraints. As we add these extensions that analysis becomes progressively more realistic but also progressively more complex. Finally, in section 7 we provide some conclusions and discuss how the insights from this simple model of the IPO decision can help guide and illuminate structural empirical studies of the IPO decision based on more realistic but also more complex versions of the model introduced in this paper.

2 What we know empirically about the decision to go public

There is a relatively large empirical/applied literature studying the decision to go public. A good starting point is the book Draho [2004], who notes that "Few events in the life of a company are as great in magnitude and consequence as an initial public offering (IPO)." (p. 1). When we use the term "going public" we refer to a decision by a private firm to list and trade their company stock on a public stock exchange, such as the New York Stock Exchange.² According to Johnson et al. [2017], the vast majority, 94%, of US firms that went public in 2016 are incorporated in Delaware and 64% of them choose to list their stock on NASDAQ and the remaining 36% list on the New York Stock Exchange (NYSE). The finance literature has documented a number of empirical regularities related to IPOs and we summarize the findings that are relevant to our theory below.

2.1 Size distribution of IPOs

Johnson et al. [2017] provide an annual report on IPOs based on Security and Exchange (SEC) filings. In 2016 there were 98 IPOs in the US, with a median value IPO proceeds was \$94.5 million. The distribution of proceeds is right skewed with a long upper tail: 21.4% of IPOs in 2016 yielded proceeds less than \$50 million, and only 9.2% yielded more than \$500 million. The largest IPOs in history include Alibaba in 2014 which raised \$25 billion, Visa in 2008 (\$17.9 billion), and Facebook in 2012 (\$12 billion).

2.2 The high cost of IPOs

IPOs almost always require the support of an *underwriter* such as an investment bank that manages the process of selling a large block of newly issued shares for the new public venture for the first time. A large IPO is typically managed by a lead underwriter who forms a syndicate of investment banks that attempt to sell large blocks of the newly issued shares to pension funds, insurance companies, while holding residual shares themselves in an attempt to manage the market impact of the IPO to avoid depressing the share price by avoiding selling too many shares too soon.

Draho [2004] illustrates the advisory and listing fees associated with a typical IPO using information from NASDAQ and NYSE. The biggest of these is the *underwriting spread* (commission), and for an illustrative \$100 million IPO, the total costs range from \$8.4 to \$8.8 million and consist mainly of a 7% underwriting spread plus additional fixed costs such as legal, accounting, and due diligence costs to prepare a prospectus for the IPO and various filing and listing fees. As Draho [2004] notes, "The fixation on a 7

 $^{^{2}}$ Thus, an IPO is distinct from sale of *private equity shares* that can be accomplished much more cheaply and informally, but with the drawback that there is likely no secondary market that enables shareholders to trade their shares unlike in the case of a publicly listed company whose shares are traded on a stock exchange.

percent spread has led many observers to suggest that collusion might be behind underwriter compensation. Adding fuel to the fire is the fact that spreads in Europe and Asia average between 2.5 and 4 percent. Even underwriters admit that spreads are high (Chen and Ritter [2000])." (p. 194).

The total costs of an IPO are significantly higher once the implicit cost of IPO *underpricing* is taken into account. This term refers to situations where the underwriter's early sales of the new shares tend to be at below market prices, resulting in higher than normal returns for the investors who agree to buy blocks of shares in the IPO shortly after they are offered. We do not focus on the details of the underwriting process or the issue of underpricing. Instead, we we regard underpricing as a temporary phenomenon that can be regarded as a component of the underwriting cost — an excess return to initial investors to compensate them for the higher risk and investment costs due to the limited track record available for companies that are just going public for the first time.

As Ritter [1987] notes, there are two main types of underwriting contracts: 1) *firm commitments* and 2) *best efforts*.³ The issuing firm and the underwriter face risk under either contract, but to the extent that the underwriter fully or partially insures the amount IPO proceeds to an issuer, part of the high underwriting costs may be justified as a compensation for the underwriter's risks, especially in firm commitment contracts. Ritter [1987] also includes underpricing as part of the total cost of an IPO, and finds that "Both components are economically significant, with total costs, expressed as a percentage of the realized market value of the securities issued, averaging 21.22% for firm commitment offers and 31.87% for best efforts offers."⁴

2.3 Why do firms go public?

This is the first question rasied in the survey by Ritter and Welch [2002] who conclude that "In most cases, the primary answer is the desire to raise equity capital for the firm and to create a public market in which the founders and other shareholders can convert some of their wealth into cash at a future date." They assert that non-financial reasons, such as publicity associated with going public, "play only a minor role for most firms: absent cash considerations, most entrepreneurs would rather just run their firms than concern themselves with the complex public market process." These conclusions are consistent with the subsequent survey by Brau [2012] who reports reasons for going public obtained from direct opinion

 $^{^{3}}$ In the former, the underwriter guarantees the net of commission proceeds from the IPO, whereas a best efforts contract specifies a mininum and maximum number of shares to be sold, an offer price, and the underwriter's promise to make a best effort to sell the minimum number of shares at the offer price. However if the underwriter fails to do so "within a specified period of time, usually 90 days, the offer is withdrawn, the investors' money is refunded, with the issuing firm receiving no money." (p. 270).

⁴One might expect that the firm commitment contracts would be more expensive to reflect the value of the insurance they provide issuers, but Ritter [1987] speculates that the higher cost of best effort IPOs may reflect adverse selection "if there is enough uncertainty about the value of the firm, an issuing firm is better off using a best efforts contract because the required underpricing if it used a firm commitment contract would be so severe." (p. 280).

surveys of a sample of 984 CFOs of private companies in an original study by Brau et al. [2006] "Only three survey questions received at least 75% agreement as an advantage of conducting an IPO: to gain financing for long-term growth (86.8%), to gain financing for immediate growth (86.8%), and to increase liquidity (82.5%)." Nearly 69% of the surveyed CFOs strongly agreed with the statement "A disadvantage of the IPO was that it made our company suddenly open to public scrutiny."

However lack of data on private firms has been a problem hindering our understanding of why firms go public. Brau [2012] notes that "Without private firm data, it is difficult to compare private and public firms to isolate the factors determining why firms go public." Ritter and Welch [2002] note that "formal theories of IPO issuing activity are difficult to test" because of a standard selection bias: "researchers usually only observe the set of firms actually going public. They do not observe how many private firms could have gone public."

Pagano et al. [1998] and Kim and Weisbach [2008] are among the few large scale econometric analyses of the decision to go public. The former study follows a panel data set of 2,181 private firms in Italy to "analyze the determinants of initial public offerings (IPOs) by comparing the ex ante and ex post characteristics of IPOs with those of private firms" (p. 27). They find that while there are very large private firms that have not gone public, the probability of going public is increasing in the size of the company. However they note that "the Italian stock market is very small relative to the size of its economy" and their conclusions for Italian firms do not generalize to other countries: "The typical Italian IPO is 8 times as large and 6 times as old as the typical IPO in the United States. As the fixed component of the direct listing costs does not differ significantly, this raises the question of why in Italy firms need such a long track record before going public." However a more surprising conclusion is that"companies do *not* go public to finance subsequent investment and growth, but rather to rebalance their accounts after a period of high investment and growth." (p. 61).

Contrary evidence was provided a decade later by Kim and Weisbach [2008] in an econometric analysis of a sample of 17,226 initial public offerings and 13,142 seasoned equity offerings from 38 countries between 1990 and 2003. They conclude that "Our results suggest first that equity offers are used to raise investment capital. Specifically, our estimates imply that R&D expenditures increase by 18.5 cents per marginal dollar of capital raised in the first year following an IPO, and by 17.8 cents per marginal dollar raised in the first year following an SEO. These figures increase to 78.0 cents per dollar raised if the changes are computed over the four-year period following IPOs and 64.3 cents for the four-year period following SEOs. These estimated expenditures are substantially, and statistically significantly, larger than the comparable numbers for a marginal dollar of internally generated cash. They also appear to be similar over alternative legal regimes. These results strongly suggest that one motive behind equity offers is to raise capital to finance investment." (p. 301).

The conflicting findings regarding the effect of IPOs on firm performance is one of puzzles in the IPO literature that we will discuss in more detail below. Overall, there seems to be agreement that IPOs are motivated by the need for capital and do spur investment and growth, but there is conflicting evidence on whether this growth is associated with greater profitability.

There has been comparatively little work studying the extent to which IPOs and SEOs are motivated by *financial constraints* such as binding borrowing constraints imposed by banks or other lenders. Bergbrant et al. [2017] provide recent time series evidence that greater availability of credit has a strong *negative* effect on the propensity to go public. "Using residual lending standards as a clean measure of aggregate loan supply and a VAR framework to aid identification, we find that a one-standard-deviation shock to lending standards results in 15% fewer IPOs. Shocks elicit strong responses from IPO-firms that are highly dependent on external capital and increase the number of withdrawals, strengthening the interpretation that the above is driven by changes in the supply of equity. Our results suggest that credit conditions are important to a well-functioning IPO market." (p. 32).

2.4 When do firms go public?

Generally it is younger, "startup" firms that go public. Jay Ritter maintains a large dataset of 8,249 US IPOs between 1980 and 2016 on his website https://site.warrington.ufl.edu/ritter/ipo-data/. Over this entire time span the median age of firms that went public is 8 years, and there seems to be an upward trend in the mean age over time: in 1980 the median age for the 71 IPOs was 6 whereas in 2016 the median age for the 74 IPOs in that year was 10.

Johnson et al. [2017] report that the median annual revenue of US companies that went public in 2016 was \$66 million. Approximately 50% of these had venture capital (VC) investments prior to their IPOs and the median such company received \$98 million in VC funding for a median of 7.7 years before going public. Surprisingly, Johnson et al. [2017] report that only 36% of the companies going public in 2016 were "profitable" (i.e. reporting positive accounting profits). Thus, perhaps not surprisingly, firms that go public are not only young, but they are also generally small. This seems to be a consistent finding in the US. For example Weinberg [1994] notes that "While the size distribution of firms undertaking IPOs varies from year to year, it typically includes many small firms (assets less than \$10 million). In 1984, virtually all IPOs were by small firms, while in 1985 and 1986, small firms conducted about half of all offerings." (p. 22).

However the conclusion that mainly "young and small" companies go public may not apply to all countries. For example the study by Pagano et al. [1998] found that the average age of the very small number (68) of Italian companies that chose to go public out of they study universe of 19,817 initially private companies was 33 years. Further the firms that went public were "twice as large as the median

potential IPO in terms of sales, employees and total assets. By contrast, the median IPO is not more profitable than the media potential IPO and is more highly levered." (p. 36).

Of course, we should not forget the other major conclusion about when most private companies go public: *never*. Some of the world's largest and oldest firms have always been private and are likely to remain so. Examples include Cargil and Koch Industries which each have over \$100 billion in annual revenue. So any theory of when firms go public must explain why only a small fraction of all firms go public and why the vast majority of firms choose to stay private, including very large firms.

2.5 IPO cycles

A related aspect of when firms go public are the rather pronounced cycles in the aggregate number of IPOs, which exhibit significant year to year time series variability. In the US the number of IPOs ranged from a high of 845 IPOs in 1996 to a low of 27 in 2008, and peaks in IPOs generally occur during peaks in the overall value of the stock market (such as measured by the Shiller PE ratio). Thus, Ritter [2013] concludes that "IPO volume is higher when stock prices are higher." (p. 128).

The existence of IPO cycles has long been recognized in the finance literature "Clustering of initial public offerings (IPOs) is a well-documented phenomenon. Starting with Ibbottson and Jaffe [2005] several studies have shown that IPOs tend to cluster both in time and in industries." (Alti [2005], p. 1105). Lowry and Schwert [2002] find that "Both IPO volume and average initial returns are highly autocorrelated. Further, more companies tend to go public following periods of high initial returns." (p. 1171). Chemmanur and He [2011] provide empirical evidence in favor the hypothesis that competition for market share prompt firms with enough sufficient internal capital to go public, which can lead to IPO cycles. Chemmanur and Fulghieri [1999] analyzed the trade-off between firms choice of going public to sell shares to numerous investors versus remaining private and funding its investments by venture capital in an asymmetric information setting. They showed that an entrepreneur with private information about the value of the firm will decide when to go public based on the magnitude of the cost of information production by investors. When sufficient amount of information about the firm has been accumulated in the public domain, it reduces outsiders information production costs and can lead to a clustering of IPOs. They showed that "hot issue markets" can occur when there is a sudden unanticipated productivity shock in an industry that cause many firms in the industry go public at the same time.

2.6 The effect of IPOs on operating performance

As we noted above, one of the major puzzles around IPOs is the possible tendency of firm operating performance to peak right before the IPO and fall thereafter. One of the first studies to document this was Jain and Kini [1994] who found a delcine industry adjusted post IPO operating performance (return on assets, operating cash flow etc.) relative to their pre-IPO level even though they also documented significantly higher growth rates of sales and investment after the IPO. They also found a positive relationship between the operating performance of the post-IPO company and the ownership share retained by the original owners. Though this could be evidence in favor of the agency theory of Jensen and Meckling [1976], because they do not observe the ownership stake of the post-IPO managers, they conclude "It is not possible, however, to determine whether the relatively superior operating performance occurs as a result of lower agency conflicts when there is higher ownership retention, as a result of entrepreneurs signaling quality with ownership retention, or for other reasons." (p. 1725).

Subsequent studies such as Mikkelson et al. [1997] also found deterioration in certain measures of the operating performance of firms that go public, but is unrelated to the change in post-IPO ownership stake held by managers: "We conclude that the changes in equity ownership that result from going public do not lead to changes in incentives that affect operating performance." (p. 306). Thus, the empirical evidence for the agency theoretic explanation of Jensen and Meckling [1976] on firm performance is mixed.

However the finding that growth accelerates before an IPO and declines afterward is puzzling and seems more robust. We would expect the opposite after an IPO, i.e. a spurt of growth from the new investment financed by IPO proceeds. It may be possible that the decline in post-IPO performance is more of a measurement issue than a real issue. In their studies of the IPO decisions of banks Rosen et al. [2005] also find that "profitability of IPO banks may decline relative to their peers after the IPO" and "evidence of rapid growth and high profitability leading up to the IPO" however they note that "since we use accounting measures of profitability, it possible that we are capturing the effects of banks manipulating their accounting data to inflate pre-IPO profit at the expense of future profitability."

However it appears that the consensus is the decline in operating performance after an IPO is not just an artifact of accounting mismeasurement or manipulation. For example the model of Clementi [2002] is devoted specifically to explaining this puzzle. He cites a number of other studies including Degeorge and Zeckhauser [1993] and Fama and French [2004] and concludes that "These studies present evidence that, for IPO firms, measures of operating performance such as ratios of Operating Earnings and Cash Flows over Book Value of Assets exhibit a sudden decline in the fiscal year in which the offering takes place, and keep on worsening for a few more years."

2.7 IPOs and dilution of ownership

Dilution refers to the decision on how large a share of the post-IPO company the original owner chooses to retain after an IPO. Levtov [2016] used data on 71 IPOs of tech sector companies between 2000 and 2008 to study the amount of equity they held just after the IPO: "On average, all founders combined owned 15%

of the company, which was worth \$100 million." Founders' share of the post-IPO company ranged from a high of 75% for Atlassian to a low of 0% for Zipcar. Another report by Bort [2014] notes that for many recent tech IPOs "Often they own less than 10% of their own companies. For instance, among the tech industry's most recent S1 forms, Aaron Levie, founder of Box, will own about 6% after the IPO. Zendesk co-founder and CEO Mikkel Svane will own about 8% after the IPO." This article looks into reasons why these founders retain such a low fraction of the company and notes that "As founders raise more funds, their share gets diluted — meaning the percentage of the company they own gets smaller and smaller. But the dollar value of the stake should be worth more: a smaller piece of a growing pie."

One might expect that the founder's share declines with the size of the IPO proceeds, but this is frequently not the case. Though most IPOs are relatively small (raising \$100 million or less) there is a thin upper tail of IPOs that raise billions of dollars, such as Facebook's IPO in 2012 which raised \$17 billion in current dollars, and the founder, Mark Zukerberg, owned approximately 30% of the company after the IPO. Another recent example is the IPO by Snap, Inc. which raised approximately \$21 billion, and the two co-founders held approximately 35% of the shares after the IPO.

There are fewer academic studies that we are aware of about dilution in shareholdings after an IPO. Foley and Greenwood [2010] study the evolution of ownership of companies in 34 countries that did IPOs between 1995 and 2006 and found that *blockholdings* (i.e. large blocks of shares owned both by the founders and other large shareholders who invested in the IPO such as mutual funds, pension funds, insurance companies, etc) are high right after the IPO (constituting on average 60% of the shareholdings) but "experience decreases in ownership concentration; these decreases occur in response to growth opportunities, and they are associated with new share issuance." (p. 1231). Pagano et al. [1998] find that Italian firm shareholders retain an average of 69% ownership at the IPO and 64% three years after the IPO. Using US data, Mikkelson et al. [1997] report 44% ownership retention, and using UK data, Brennan and Franks [1997] report a 35% ownership retention.

There is even a less studied question of what share of the IPO proceeds the private owner takes out in cash rather than reinvests in the new founded public company. When we use the term "cash out" it is important to distinguish from a direct use of the IPO proceeds for consumption or other investment, versus a subsequent decision by the owner to sell some of his shares after the IPO. For example *CNN.Money* reported in 2012 that Mark Zuckerberg planned to sell 30.2 million of the 534 million shares he owned after the Facebook IPO. Technically, we do not consider such subsequent stock sales by the owner to be part of the "cash out" which we assume can only occur at the time of the IPO.

Wahba [2008] notes that "Paying off debt has always been part of what many IPO proceeds have been earmarked for. But with the credit crisis making borrowing more expensive, it has become even more pronounced and more crucial." This article noted examples where founders took large shares of IPO proceeds in cash: "the Blackstone Group, which floated shares last year, took some cash out, including the co-founder and senior chairman, Peter Peterson, who got \$1.92 billion in IPO proceeds. The firm said its IPO included raising about \$3 billion in fresh capital and imposed some restrictions on vesting. Blackstone's shares are currently trading about half of their IPO price." Overall, the success of an IPO depends on the credibility of the firm's signals about how it will use the IPO proceeds, since as Das [2008] notes "If a company does not adequately explain how it is going to use the money in its regulatory filings, the chances are investors will not touch the stock."

Academic studies such as Brau et al. [2007] find that insider selling after an IPO is related to poorer long-run performance suggesting that these sales may indicate agency issues or adverse selection problems. Ang and Brau [2003] documented that when insiders objective is to cash out in the IPO, they retain less of their shares at the original filing, and increase secondary shares through amendments, which may be part of a confounding/concealment strategy to prevent outside shareholders from drawing an unfavorable impression about the post IPO value of the firm.

There are also studies that focus on the diversification of the owner *prior* to undertaking an IPO to see if the increased liquidity and potential for diversification is a motivation for going public. Bodnaruk et al. [2008] used data from Swedish IPOs to show that less diversified owners have higher incentives to go public; a one-standard-deviation increase in the diversification measure above its mean results in a 2.28% reduction in the probability of going public. They also documented that less diversified shareholders are more willing to sustain higher costs of doing an IPO (including via underpricing) in exchange for the enhanced ability to diversify their wealth by going public.

2.8 IPOs and product market

IPO may have competitive effects, such as enabling a firm to raise funds for investments that may be helpful in deterring entry in imperfectly competitive product markets. Several studies have documented the role of the product market competition on the decision to go public. Chemmanur et al. [2009] used the Longitudinal Research Database from the US census bureau to track firms before and after going public. They conclude that "First, firms with larger size, sales growth, total factor productivity (TFP), market share, capital intensity, access to private financing, and high-tech industry membership are more likely to go public. Second, firms operating in less-competitive and more capital-intensive industries, and those in industries characterized by riskier cash flows, are more likely to go public. Third, firms with projects that are cheaper for outsiders to evaluate, operating in industries characterized by less information asymmetry, and having greater average liquidity of already listed equity, are more likely to go public. We also show that, as more firms in an industry go public, the concentration of that industry increases in subsequent years." They also find that "that although TFP and sales growth exhibit an inverted U-shaped pattern (with

peak productivity and sales growth occurring in the year of IPO), sales, capital expenditures, employment, total labor costs, materials costs, and selling and administrative expenses exhibit a consistently increasing pattern in the years before and after the IPO." (p. 1905).

Chod and Lyandres [2011] hypothesized that a strategic benefit of being a public and diversified firm is that it helps management to take more risky product market strategies that can boost their competitive position. They find empirical support for this prediction, and that the decision to go public is related to the degree of competitive interaction and demand uncertainty.

2.9 IPOs, financial constraints, and firm growth

Despite the conflicting results about post-IPO operating performance, there is general agreement that IPOs do lead to substantial capital infusions that enable firms that go public to invest and grow. For example Kenney et al. [2012] follows 2,766 US companies that went public between 1996 and 2010. They compare sales and employment before and after the IPO and find the employment is 36% higher and sales is 65% higher 3 years after the IPO and 5 years after the IPO employment and sales have increased by 60% and 85%, respectively. These appear to be high returns to investment given the average per firm IPO proceeds were \$162 million: "on average every [new] job [created] required an investment of \$200,000." Collectively they find that the companies that went public over this period added "2.272 million employees after the IPO average increase of 822 employees per firm. In dollars of 2011 purchasing power, their combined annual revenue grew from \$1.32 trillion prior to the IPOs to \$2.58 trillion in fiscal 2010." (p. 20).

There has a huge amount of research on the negative effect of financial constraints on firm growth in the finance and economics literature. The earliest work was based on neoclassical investment models and the "Tobin-q" theory of investment that largely ignored financial constraints. However Whited [1992] notes that "tests of the q-theory of investment have found little explanatory power for q, have implied implausibly slow capital stock adjustment speeds, and have been outperformed by simple ad hoc accelerator models." (p. 1425). She develops and empirically estimates a dynamic model of finance and investment that recognizes "that small firms with low liquid asset positions have limited access to debt markets, presumably because they lack the collateral necessary to back up their borrowing." (p 1426). Whited [1992] structurally estimates an Euler equation model of optimal investment by firms and finds that "Including the effect of a debt constraint greatly improves the Euler equation's performance in comparison to the standard specification. When the sample is split on the basis of two measures of financial distress, the standard Euler equation fits well for the a priori unconstrained groups, but is rejected for the others." (p. 1925). She concludes "that any attempt to understand investment in the aggregate must account for firms' differential access to capital markets-in particular, debt markets." (p. 1450).

In the industrial organization literature, there are two broad empirical regularities of the firm as noted in Cooley and Quadrini [2001]: dynamics of firms (growth, job reallocation, and exit) are negatively correlated with the initial size of the firm and its age. Their paper provides a dynamic model of firm growth that incorporates financial frictions that can provide qualitative explanations for some of the observed empirical regularities. Financial frictions arise in their model in the form of costs of issuing equity and default costs that cause the Modigliani-Miller Theorem to fail, so that debt and equity are no longer perfect substitutes and the investment choice of the firm depends on the amount of equity it owns. They conclude that "Existing models of industry dynamics that abstract from financial-market frictions are unable to account simultaneously for the dependence of the firm dynamics on size and age." but their model constitutes "a first step toward the study of the importance of financial-market frictions for the dynamics of the firm." (p. 1303).

To our knowledge, there there has been limited work on extending the structural empirical approaches pioneered by Whited [1992] and Cooley and Quadrini [2001] to consider whether IPOs represent an alternative means of financing firm growth when firms face binding borrowing constraints in debt markets. We have already noted recent reduced-form evidence by Bergbrant et al. [2017] that shows that tightening in the credit market does significantly spur the rate of IPOs. However we are unaware of theoretical or structural econometric models that provide a micro-level underpinning for these reduced-form correlations other than Clementi [2002], who formulated a dynamic model of going public decision where the firm is subject to productivity shocks. However the focus of his model is to explain the post-IPO decline in profitability and rates of return that we summarized in section 2.6.

2.10 The secular decline in IPOs in the US: excessive regulation?

Though we noted that the cyclic behavior of IPOs is well known, there is more recent concern about a possible *secular decline* in IPOs, at least in the US. For example a recent article by Macey [2017] notes that "The number of public companies has shrunk by more than one-third during a time when the U.S. economy has more than doubled in size. In 1997, there were 9,113 public companies in the U.S. At the end of 2016, there were fewer than 6,000." Ritter [2013] notes that despite year to year variation in IPOs, there seems to be a permanent decline in the number of new IPOs "From 1980-2000, an annual average of 310 operating companies went public in the United States. During 2001-12, on average, only 99 operating companies went public. This decline occurred in spite of the doubling of real gross domestic product (GDP) during this 33-year period. The decline was even more severe for small-company initial public offerings (IPOs), for which the average volume dropped 83 percent, from 165 IPOs a year during 1980-2000 to only 28 a year during 2001-12." (p. 123, 125).

If this secular decline does exist, there is further debate was to how much we should be concerned

about it and whether a decline in IPOs is only symptomatic of larger structural issues with the US economy that Decker et al. [2016] and Decker et al. [2017] refer to as "declining dynamism" "Evidence of declining entrepreneurship and labor market fluidity has captured wide interest among researchers and policymakers. Startup rates and other measures of young firm activity have declined since the 1980s, with accelerated slowdowns in high-growth young firm activity since 2000. Gross job and worker flows have declined over the same period including marked drops since the early 2000s. These patterns are particularly notable in the High Tech sector, which saw rising dynamism during the 1990s before declining sharply after 2000." According to Decker et al. [2017] we should be concerned about this trend, since "declining business dynamism has not been benign for American living standards but, instead, is closely related to slowing productivity growth."

Others have suggested that at least for the narrower question of the decline in IPOs, excessive regulation may be responsible: "The conventional wisdom is that the main culprits are a combination of heavy-handed regulation, especially the Sarbanes-Oxley (SOX) Act of 2002, a decline in analyst coverage of small firms, and lower stock prices since the 2000 technology bubble burst." (Ritter [2013], p. 125). This view has stimulated policy responses such as the Jumpstart Our Business Startups (JOBS) Act passed in 2012, which facilitates startups in a number of ways by easing various securities regulations as well as calls to repeal or roll back the SOX act, which "requires external audits of the internal control systems of publicly traded companies to ensure that their financial reports are accurate." However Ritter [2013] notes that small firms were exempted from SOX regulations in 2007, yet "small-company IPOs should have rebounded after 2007" and "evidence from Europe suggests that heavy-handed regulation has not been the prime deterrent of small-company IPOs." and this leads him to conclude that "SOX has not been the primary reason that the volume of small-company IPOs has been low for more than a decade in the United States, although this does not mean that heavy-handed regulation has had no effect on IPO volume" (p. 126).

Thus, Ritter [2013] suggests that "lack of profitability of small companies" (perhaps due to greater market power exercised by increasing concentration of very large firms, which is part of the decreasing dynamism that Decker et al. [2017] highlight) may have more to do with the decline in IPOs than excessive regulation. Further, he speculates that "I do not think that the JOBS Act will result in a flood of companies going public. The main reason why fewer small companies have been going public is that they are finding it difficult to earn a profit. The JOBS Act does little to solve this problem. Nor do I think that noticeably higher economic growth and job creation will result from the JOBS Act." (p. 142). Thus, Ritter [2013] is not excessively concerned about the secular decline in IPOs, if this trends really exists: "In summary, I do not know what the optimal level of IPO activity is in the United States or any other country, nor do I think that it should necessarily be the same now as it once was. I believe that a long-term change has

been occurring in which getting big fast is now more important than was once the case, at least in certain industries. Because merging is sometimes the most efficient way of getting a successful new technology to market quickly, I do not view the increase in trade sales and the decrease in IPO activity as necessarily alarming." (p. 143).

2.11 Dividend smoothing by private and public firms

Finally, we summarize a significant literature on "dividend smoothing" by public firms, since it is relevant to our theory of IPOs, which suggests that public firms, to the extent that they should be maximizing the equity value of shareholders and thus maximizing expected dividend, should *not* be engaging in dividend smoothing. However Wu [2015] states that "Dividend smoothing is one of the oldest and most puzzling phenomena in corporate finance" (p. 1). The earliest work by Lintner [1956] provided survey evidence from managers of public companies that they believe their shareholders put a high value on stable dividend payments. Brav et al. [2005] also used interviews with CFOs to conclude that public firms are willing to raise external capital or defer or forego attractive investments to avoid cutting dividends. Leary and Michaely [2011] in a cross-sectional econometric analysis using CRSP data from 1985 to 2005 "find that younger, smaller firms, firms with low dividend yields and more volatile earnings and returns, and firms with fewer and more disperse analyst forecasts smooth less. Firms that are cash cows, with low growth prospects, weaker governance, and greater institutional holdings, smooth more." (p. 3197). Using a second time series data set on CRSP data going back to 1925 they find that "over the past 80 years, dividend smoothing has been steadily increasing, even before firms began using share repurchases in the mid-1980s." (p. 3243).

In an international study using 2219 firms from 24 countries, Javakhadze et al. [2014] find that "firms with highly-concentrated ownership structure and strong corporate governance smooth dividends less." (p. 200). This result is also puzzling since we might intuitively expect public firms with highly concentrated ownership structures to behave more like privately owned firms.

Our theory is based on the hypothesis that private firms have a much greater incentive to dividend smooth than public firms, since we model private firms as discounted expected utility maximizers and we will show in sections 4 and 6 that this causes private firms to value smoothed dividend streams more highly than variable dividend streams. Michaely and Roberts [2011], using accounting data from approximately 2.1 million public and private firms in the UK from 1993 to 2002, compare the dividend policies of public and private firms and find "a great deal of heterogeneity in dividend policies across public and private firms." (p. 741). However rather surprisingly, they conclude that "private firms smooth dividends significantly less than their public counterparts" and speculate "the scrutiny of public capital markets plays a central role in the propensity of firms to smooth dividends over time. Public firms pay relatively higher

dividends that tend to be more sensitive to changes in investment opportunities than otherwise similar private firms." (p. 712).

These puzzling and counterintuitive empirical findings have given rise to several theories to explain why public firms find it so important to smooth dividends for their shareholders, including Fudenberg and Tirole [1995] who treat dividend smoothing as a type of manipulation by firm managers due to "concern about keeping their position or avoiding interference" (p. 75) and Guttman et al. [2010] who posit that "The manager cares about the short-term stock price in addition to its long-term (intrinsic) value" and "Linking the managers compensation to short-term stock price induces her to raise the dividends in order to signal higher earnings, resulting in underinvestment relative to the first-best level." (p. 4457).⁵

2.12 Comments

This short survey represents our understanding of the relevant empirical literature on IPOs. It summarizes the "empirical facts" that we want the reader to keep in mind when evaluating the predictions of our theory of why and when firms go public. Of course any theory is a vast simplification of reality and not all of our assumptions will be empirically "realistic". In particular, we make assumptions about the objectives of owners of private firms and managers of public firms that may or may not be consistent with empirical findings on dividend smoothing discussed above. After presenting the theory and making clear the nature of its qualitative predictions and implications, in the conclusion we will discuss which of these predictions are consistent with the empirical facts that we have summarized above and which are not. We will argue that the simple theory we provide below can be extended, though at the risk of additional complexity, to accomodate many of the empirical facts of IPOs that we have summarized in this section, and provide a framework for evaluating policies that affect IPOs such as innovations that reduce the high cost of IPOs and regulatory changes such as SOX and the JOBS Act discussed in section 2.10.

3 Investment and Financial Policy for a Publicly Held Firm

We start our theoretical analysis by deriving the investment and financing strategy of a public firm using backward induction. This analysis will yield formulas for the stock market valuation of a public firm that provides the continuation value that plays a key role in the private firm owner's decision of whether to remain a private firm or go public via an IPO that we study in section 4. We assume that being public is

⁵Recent structural econometric work by Zakolyukina [2017] also provides evidence in favor of the hypothesis that a CEO's "compensation and career path depend on the stock price, thus inducing him to work hard but also to misstate earnings to manipulate the stock price". Using data on on financial accounting restatements she concludes that "although the probability of manipulation being detected is low, the perceived penalty upon detection for sizable mis- statements is substantial" and "the average magnitude of manipulation is higher for small firms and firms with low leverage."

an absorbing state: a more complex analysis would be required if we allow a firm to transit back and forth between public and private ownership status.

We start our analysis assuming the firm does not have access to capital markets, that is, it is not allowed to borrow. In this case, the only way for the firm to finance its desired investments is via retained earnings. Then we successively introduce perfect debt and equity markets and show how this affects the investment and financial policy of the firm. By "perfect" we mean debt and equity markets where there are no constraints on the amount the firm can borrow or transactions costs in raising additional equity capital via issuance of additional shares. Finally, we assume that equity markets obey the efficient markets hypothesis, so that the firm's stock market valuation equals the expected present value of its dividend stream.

To illustrate these points in the simplest possible dynamic setting, we ignore labor and material inputs to production, managerial "effort" and other details of the firm's operations such as marketing and research and development, and assume the net cash flow that a firm generates is a strictly concave function f(k)of its capital stock k, where this capital could either be physical capital (e.g. plant and equipment) or intellectual capital (e.g. patents and technological know-how). We assume that capital is of the "puttyclay" variety so that once the capital is installed, the firm cannot liquidate or resell it. We make the standard assumption that the capital stock depreciates at a constant rate of $\delta \in (0, 1)$ per period, and the firm can replenish or grow its capital stock via investment at a constant price of \$1 per unit of new capital installed.

Let V(k) denote the value of a publicly held firm when its capital stock is $k \ge 0$. We assume that a publicly held firm's objective is to maximize the value of its equity, which in an efficient capital market is the discounted value of dividend payments to its shareholders. The Bellman equation for the firm is given by

$$V(k) = \max_{0 \le I \le f(k)} [f(k) - I + \beta V(k(1 - \delta) + I)],$$
(1)

where $\beta = 1/(1+r)$, and *r* is the market rate of return at which the firm's dividend stream is discounted. If f(0) = 0, then it is clear that V(0) = 0, since when the firm has no capital investment, it generates no cash returns, and thus it cannot invest any more funds, and therefore will not receive any future cash flows from which it can pay out dividends in the future. Implicit in the formulation (1) is a one period lag before new investment is operational.

Assume that the "Inada condition" holds for f(k), so the marginal return to investment f'(k) approaches infinity, $\lim_{k\downarrow 0} f'(k) = \infty$ and $\lim_{k\uparrow\infty} f'(k) = 0$. Since the return to investment becomes unboundedly high when the capital stock is sufficiently small, it is reasonable to conjecture that the firm's optimal investment policy has three different regions: 1) an initial region $[0,\underline{k}]$ where the firm pays no dividends and devotes all cash flows to investment, 2) an intermediate region $[\underline{k},\overline{k}]$ where the firm invests and pays dividends, and 3) a final region (\overline{k},∞) where the firm has "excess capital" and so it does not invest and

pays out all cash flow in the form of dividends.

In the intermediate zone where the firm invests and pays dividends, the firm invests just enough to immediately jump to an optimal target or "steady state" capital stock k^* , which is the solution to problem

$$k^* = \underset{k \ge 0}{\operatorname{argmax}} \beta \frac{f(k) - \delta k}{(1 - \beta)} - k \tag{2}$$

Thus k^* is the optimal steady state capital stock that maximizes the discounted present value of the firm, net of the cost of the initial investment k. From the first order condition to (2) we see that k^* is given by

$$k^* = f'^{-1} \left(1/\beta - 1 + \delta \right). \tag{3}$$

where f'^{-1} is the inverse of the marginal return function, f'(k), which is invertible due to our assumption that *f* is strictly concave, which implies that f''(k) < 0. Since $\beta = 1/(1+r)$ where r > 0 is the one period market interest rate, then we can rewrite the first order condition for the optimal steady state capital stock k^* as follows

$$f'(k^*) = r + \delta \tag{4}$$

and observe that this is identical to the equation for the Golden rule steady state capital stock in the neoclassical growth model, see Phelps [1966]. The intuition for condition (4) is that a necessary condition for the steady state capital stock k^* to be optimal is that the marginal return to capital must equal the sum of the 1) depreciation of capital, δ , and 2) the opportunity cost of investment, *r*.

We can show that starting from any initial capital stock $k_0 > 0$, the dynamical system for the capital stock given by the law of motion $k_{t+1} = k_t(1-\delta) + I(k_t)$ is globally stable with a single limit point k^* . However the dynamical system also has a second steady state solution $k^* = 0$, since it is easy to see that if we start with initial capital $k_0 = 0$ then, since I(0) = 0 it follows that $k_t = 0$ for all $t \ge 1$. Thus, the firm needs positive initial capital investment to get started.

These results follow from the form of the optimal investment rule I(k) from the firm's dynamic programming problem (1).

$$I(k) = \begin{cases} f(k) & \text{if } k \in [0, \underline{k}) \\ k^* - (1 - \delta)k & \text{if } k \in [\underline{k}, \overline{k}] \\ 0 & \text{if } k \in (\overline{k}, \infty). \end{cases}$$
(5)

It is easy to see that \overline{k} is given by

$$\overline{k} = \frac{k^*}{(1-\delta)} \tag{6}$$

and \underline{k} is given by

$$f(\underline{k}) = k^* - (1 - \delta)\underline{k} \tag{7}$$

These values of \underline{k} and \overline{k} ensure that the optimal investment function I(k) is a continuous function of k. Since there is no borrowing or other cash holdings in the model, the optimal dividend function is given by

$$D(k) = f(k) - I(k).$$
(8)

We now introduce an additional assumption on the production function and then summarize these results in Theorem 0 below.

Assumption 0 The production function f(k) is strictly increasing, concave and satisfies f(0) = 0 and the Inada Condition, i.e. $\lim_{k \downarrow 0} f'(k) = +\infty$. In addition the following finite period reachability condition holds

$$\liminf_{k \ge 0} \{t | g(k,t) > \underline{k}\} < \infty \tag{9}$$

where the sequence of functions g(k,t) is defined recursively by g(k,0) = k and

$$g(k,t) = g(k,t-1)(1-\delta) + f(g(k,t-1)), \quad t = 1,2,\dots$$
(10)

Thus, g(k,t) is the amount of capital the firm could accumulate if it started with initial capital stock k at period t = 0 and reinvested all profits in periods 1, 2, ..., t. The finite period reachability condition guarantees that the firm can reach the capital threshold \overline{k} in a finite amount of time from an arbitrarily small initial capital investment.

Theorem 0: Suppose Assumption 0 holds. Consider the optimal investment and dividend policy of a publicly held firm that does not have the option of borrowing, the solution to which is given by the Bellman equation (1). The optimal investment policy is given by the function I(k) in equation (5) and the optimal dividend policy is given in equation (8) where the constants k^* , \underline{k} and \overline{k} are given in equations (4), (7) and (6), respectively. Under the optimal policy, the firm will reach the optimal steady state capital stock k^* in a finite number of periods starting from any positive initial capital stock k, so $\lim_{k\downarrow 0} V(k) > 0$. However if k = 0 and f(0) = 0, the firm will remain forever in a non-investment, no-dividend absorbing state, so V(0)=0.

The proof of Theorem 0 is provided in the appendix. A key to the proof is to show that V is almost everywhere differentiable so that in the intermediate region $k \in [\underline{k}, \overline{k}]$ in (5) where the firm both invests and pays dividends, its optimal investment I(k) satisfies the following first order or *Euler equation*

$$1 = \beta V'(k(1 - \delta) + I(k)).$$
(11)

Substituting the optimal investment rule I(k) into the right hand side of the Bellman equation (1) and differentiating with respect to k, making use of the Envelope Theorem, we have

$$V'(k) = f'(k) + (1 - \delta)\beta V'(k(1 - \delta) + I(k))$$

= $f'(k) + (1 - \delta)$

where we used the fact that the Euler equation (11) holds for $k \in [\underline{k}, \overline{k}]$. The Envelope Theorem (12) implies that in the unconstrained region where $k \in [\underline{k}, \overline{k}]$ and investment is in the interval (0, f(k)), V(k) is given by

$$V(k) = f(k) + (1 - \delta)k + C$$
(12)

for some constant *C* when $k \in [\underline{k}, \overline{k}]$. Notice that at $k = k^*$ the firm generates a perpetual dividend stream of $f(k^*) - \delta k^*$ so this implies that

$$V(k^*) = \frac{f(k^*) - \delta k^*}{(1 - \beta)}.$$
(13)

So using the other formula for V(k) from equation (12), this implies that the unknown constant C is given by

$$C = \frac{\beta [f(k^*) - \delta k^*]}{(1 - \beta)} - k^*.$$
(14)

Thus, we can see that *C* equals the optimized right hand side of the net gain from initial investment in equation (2) which determined the optimal steady state capital stock value k^* . Thus, the value of the firm in the interval $[\underline{k}, \overline{k}]$ is this optimized value, plus $f(k) + (1 - \delta)k$. The intuition for this formula is that once the firm is in the interval $[\underline{k}, \overline{k}]$, its investment $I(k) = k^* - (1 - \delta)k$ will enable it to achieve the optimal steady state capital level k^* in the following period. So it follows that V(k) equals the net dividends this period, $D(k) = f(k) - I(k) = f(k) - k^* + (1 - \delta)k$ plus the present value of all future dividends in all subsequent periods $\beta [f(k^*) - \delta k^*]/(1 - \beta)$ where this period's investment has enabled the firm to achieve the optimal steady state capital stock k^* .

Now we need to verify that the optimal investment rule I(k) for $k \in [\underline{k}, \overline{k}]$ really is the formula we conjectured to hold in region 2, $I(k) = k^* - (1 - \delta)k$. To show that this is correct, we need to show that this satisfies the Euler equation (11). Using the closed form solution for V(k) in equation (12) we can rewrite the Euler equation as

$$1 = \beta \left[f'(k(1 - \delta) + I(k)) + (1 - \delta) \right]$$
(15)

Solving this equation for I(k) we can see that

$$I(k) = f'^{-1} (1/\beta - (1-\delta)) - (1-\delta)k$$

= $k^* - (1-\delta)k$

which does indeed match the formula we conjectured in equation (5). In the appendix we verify that the formulas for optimal investment in the other two regions also hold and derive closed-form expressions for the value function V(k) in these regions.

Theorem 0 tells us that the optimal investment policy is for the firm to invest all profits back into the firm and pay no dividends when the firm is sufficiently small, i.e. for $k \in (0, \underline{k})$, where $\underline{k} < k^*$ is the lower boundary of the "linear investment region" where the firm has enough accumulated capital to jump to the





optimal steady state capital stock k^* in a single period. In addition, as long as $k > \underline{k}$, the firm also has enough surplus profits to also pay dividends to its shareholders. This implies that there are zero dividends until the first period where k exceeds \underline{k} , then a "partial dividend" in that period equal to $f(k) + (1-\delta)k - k^*$, followed by an infinite stream of dividends equal to $f(k^*) - \delta k^*$.

Now consider a firm that starts out with an arbitrarily small initial investment in capital k_0 . The firm will reinvest all the cash flow from this very small initial investment and keep doing that until the capital stock first exceeds <u>k</u>. How long will this take as a function of k_0 ? The finite period reachability condition (9) guarantees that the time required to reach <u>k</u> will be finite, no matter how small k_0 is, provided k_0 is positive. This implies that the right hand limit of the value function is positive

$$\lim_{k \downarrow 0} V(k) > 0. \tag{16}$$

whereas if f(0) = 0, we know from Theorem 0 that V(0) = 0. Thus, we conclude that there is a *discontinuity in the value function at* k = 0 and this discontinuity arises naturally from the restriction that the firm is not able to "get off the ground" until at least some arbitrarily small initial investment is made in it.⁶

Figure 1 plots the optimal investment and dividend rules for the case $f(k) = \sqrt{k}$. We see that optimal investment intersects the black "replacement investment" line (i.e. the line δk) exactly at k^* , the optimal steady state capital stock level, which equals 25 in this example. The level of optimal investment at the steady state is $\delta k^* = 1.25$, which of course is just enough to offset the corresponding depreciation in capital.

⁶Of course, the finite period reachability condition may not be a particularly realistic assumption in practice: there may be fixed setup costs that must be incurred to get a firm "off the ground" and in such cases, we would expect that V(k) = 0 for all k below the minimal fixed costs that are necessary to get the firm off the ground. However the theoretical interest this is an example of a continuous state dynamic programming problem where the value function has a discontinuity. Normally we expect that small changes in "initial conditions" should only lead to small changes in payoffs, but when the finite period reachability condition is satisfied, an arbitrarily small initial investment in the firm leads to a discontinuous jump in the value of the firm — i.e. it results in an abitrarily high rate of return from this initial investment.



Figure 2: Simulated growth of a public firm, $k_0 = 0.00001$, $f(k) = \sqrt{k}$

Figure 2 illustrates the dynamics for investment, dividends and the capital stock for a public firm starting with a tiny initial endowment of capital, $k_0 = 0.00001$. For the first 19 periods the firm invests all of its cash flow and pays no dividends in order to reach the steady state capital stock $k^* = 25$ as quickly as possible. As soon as k exceeds \underline{k} , in period 20, the firm starts to pay dividends and reduce its investmena. In period t = 20 the firm undertakes a final investment that enables it to reach the steady state capital stock $k = k^*$ and thereafter investment equals δk^* and dividends equal $f(k^*) - \delta k^*$.

3.1 Extending the model to allow non-concave production functions

In the previous section we were able to derive essentially a closed form solution for the optimal investment strategy of the firm for a general case of *concave* cash flow production functions f(k). In this section we extend the model to consider *non-concave* production functions. Figure 3 plots a pair of non-concave production functions formed by grafting logistic "S-curves" on to the basic concave production function we considered in the previous section. That is, the figure plots production functions of the form

$$f(k) = \sqrt{k} + \theta_1 \left[\frac{\exp\{(k - \theta_2)/\theta_3\}}{1 + \exp\{(k - \theta_2)/\theta_3\}} \right],$$
(17)

where $\theta_2 = 80$ is a "location parameter", $\theta_3 = 10$ is a "scale parameter" that determines how steep the S-curve is, and θ_1 is a "height parameter" that determines the overall productivity. Figure 3 plots two production functions, one is for a "productive firm" where $\theta_1 = 10$ and the other is for a "less productive firm" with $\theta_1 = 2$.

The reason we believe non-concave production functions are potentially interesting is because they can enable us to model *growth stages* of firms. We can imagine a firm starting out with little initial capital and investing in a "first stage technology" that is concave, such as $f(k) = \sqrt{k}$. However after it makes

Figure 3: Non-concave production functions f(k)



its investment in its first stage technology and grows sufficiently large, the firm may be able to continue to invest in a "second stage" technology that could potentially be far more productive than its first stage technology. This second stage technology is represented by the second additive S-curve component in the production function in equation (17) and in figure 3. To reach this higher level of production and cash flow, the firm may need to undertake significant, large fixed investments that initially do not have high returns (high marginal product of capital, f'(k)) but after sufficient investment the firm can enter an *increasing returns to scale region* where f''(k) > 0 before again returning to a concave region after sufficient capital has been invested and the firm has more or less fully mastered and exploited its second stage technology.

As we noted in the previous section, the simple theory with concave production functions leads the firm to grow until it reaches the Golden rule capital stock k^* satisfying equation (4). When the production function f is concave, it is evident that there is *only one steady state*, *Golden rule solution*. However it should be clear from figure 3) that if the firm's production function is no longer concave, *there is the possibility of multiple steady state Golden rule solutions*. Which of these Golden Rule steady states will firm end up at? In addition, will the firm's investment still retain the form given in equation (5) if its production function is not concave?

Figure 4 provides the answer to this question. It plots the numerically calculated optimal investment strategies corresponding to the two production functions plotted in figure 3. The first observation is that despite the non-concavity of the production functions, the optimal investment rules I(k) still take the three region form that we illustrated in equation (5) in the concave case. That is, there are still a pair of thresholds $(\underline{k}, \overline{k})$ such that: a) the firm reinvests all of its cash flow for $k \leq \underline{k}$, b) the firm does no further *net investment*





if $k > \overline{k}$, and c) in the interval $[\underline{k}, \overline{k}]$ investment decreases linearly. Further, there is an optimal steady state capital stock, k^* , where the Golden rule relation (4) is satisfied.

The left hand panel of figure 4 plots the optimal investment rule for the firm with the productive second stage technology (i.e. $\theta_1 = 10$). In this case the optimal steady state capital stock $k^* = 108$ is in fact the largest steady state Golden rule solution. Comparing this to figure 1, we see that the steady state size of the firm is more than four times larger due to the presence of this attractive second stage investment opportunity. However the right hand panel of figure 4 shows that for the firm with the less productive second stage investment opportunity, it determines that the additional cash flow is not sufficiently high to justify investing to reach it, given that approximately the same level of capital investment would be required. This firm decides to forgo the second stage investment opportunity and thus it ends up at an optimal steady state capital stock k^* that is very near the value $k^* = 25$ that was optimal for the firm with the concave \sqrt{k} production function. Thus, for this firm the optimal steady state capital stock k^* is the *lowest* of the steady state Golden rule solutions.

Figure 5 presents simulated trajectories for the two firms, with each starting from an initial capital stock of $k_0 = 0.1$. The qualitative features of the growth in capital and output and the trajectories for investment and dividends are the same in both cases. Both firms pay no dividends and reinvest all cash flows back into the firm to grow the capital stock as quickly as possible when $k < \underline{k}$, and only begin to pay dividends when $k \ge \underline{k}$ as they steadily reduce investment as they approach their respective optimal steady state capital stocks, k^* . As we noted above, k^* is more than four times larger for the firm with the more productive second stage investment opportunity than the one with the less attractive opportunity, and thus it takes the former firm twice as long to reach its steady state capital stock k^* . However the delay is worth it as the the steady state output (plotted by the green curves in the figures) is more than 4 times as large, and so are its steady state dividends.



Figure 5: Simulated growth paths for firms with non-concave production functions f(k)

From figure 4 it is tempting to conjecture that the general form of the optimal investment rule is the same for a non-concave production function f as the general characterization for concave f that we established in Theorem 0 in the previous section in equation (5). Unfortunately in the non-concave case things are more complicated as illustrated in figure 6. We see that in this case there are *two "optimal" steady state Golden rule capital stocks* k^* . There is a low steady state at $k^* = 24.9797$ (nearly the same as for the concave example in the previous section where $f(k) = \sqrt{k}$), as well as a high steady state $k^* = 222.24$. The domain of attraction for the low steady state k^* is (0,83), whereas the domain of attraction for the high steady k^* is $[83,\infty)$. That is, if the firm starts out with sufficiently little initial capital, it will not be optimal for the firm to invest enough to reach the high steady state Golden rule capital stock at $k^* = 222.24$, and instead it is optimal for the firm to converge to the low steady state value at $k^* = 24.9797$. Essentially, the level of delay and investment necessary to reach the higher steady state k^* (which is also significantly more profitable with a value of $v(k^*) = 381$ compared to a value at the low steady state of $v(k^*) = 79$) makes it uneconomic for the firm to try to reinvest its retained earnings to grow enough to reach this higher steady state capital stock and profitability level.

However if the firm were endowed with sufficient initial capital, k > 83, then there is far less delay in reaching the higher optimal steady state capital stock and it is optimal for such a firm to reinvest all of its profits and pay no dividends until it reaches this higher steady state capital stock $k^* = 222.24$. This example highlights an interesting paradox and constraint on growth created by the firm's inability to borrow. Further, suppose the firm could borrow any amount it desired at the time the firm was initially established. As we will show in section 3.2, the firm will want to borrow an amount that will immediately put it at one of the steady state capital stocks, k^* . Suppose that the firm can borrow at the market rate of interest $r_m = 1/\beta - 1 = 0.05$. Then if the firm borrowed enough to reach the low steady state, its equity



Figure 6: More complex optimal investment policy for a non-concave production function f(k)

value is

$$E(k^*) = V(k^*) - k^*(1 + r_m) = 79 - 24.9797 * (1.05) = 52.77$$
(18)

whereas if the firm borrowed 222.24 to reach the higher steady state k^* its equity value would be

$$E(k^*) = V(k^*) - k^*(1 + r_m) = 381 - 222.24(1.05) = 149.75.$$
(19)

Thus, this provides a stark illustration of how borrowing constraints and the lack of initial capital can lead an optimizing firm to converge to lower "suboptimal" steady state outcome. We put this in quotes since the lower steady state *is* fully optimal for the firm that has insufficient initial capital and faces borrowing constraints, but it not optimal compared to a firm that does have the option to borrow. Roughly speaking, the inability to borrow causes the firm with limited initial capital to forgo undertaking an investment opportunity that could nearly triple the net of debt return to a firm that did have access to capital markets. In addition, we will see that even if firms can borrow, if they still face *borrowing constraints* they may still forgo attractive investment opportunities. It will become a key part of our explanation for why firms choose to go public via an IPO.

It is possible to extend the model in other directions to allow for other types of realistic non-convexities in the model such as the case where investment projects are lumpy — i.e. ones that require fixed initial downpayments or can only be undertaken in discrete chunks. Our model so far has assumed that capital is a perfectly divisible commodity — the firm is free to choose any level of investment it desires subject to its budget constraint. However once we allow for additional nonconvexities we need to pay closer attention to the method by which the firm finances its investments, especially in situations where the firm must come up with a fixed downpayment such as to acquire a patent or new technology that will enable it to proceed and undertake further continuous investments in what we have called a "second stage technology" especially if the firm's level of cash flow in any given period is insufficient to pay this fixed investment fee. To handle this situation, we need to introduce an additional state variable into the model, *liquid capital*, that represents cash holdings that the firm must acquire to finance large lumpy fixed investment costs. The liquid capital can be considered a type of "savings" that the firm must carry due to a *cash in advance constraint* that new investment must be financed by cash payments in situations where the firm is unable to borrow. Liquid capital is also important as a "buffer stock" against unexpected productivity shocks, so we will defer further discussion of this case until we extend the model to allow for stochastic productivity shocks, where there will be an obvious role for liquid capital.

3.2 Debt financing with perfect capital markets

Intuitively, a sufficiently small firm can accelerate its growth and market value if it has access to capital markets that allow it to borrow any desired amount. We can see from the solution to the firm's problem where the firm does not have the option to borrow that there is a liquidity constrained region $k \in [0, \underline{k})$ where the firm pays no dividends and invests all of its cash flow. Suppose that the firm can borrow unlimited amounts *b* at an interest rate *r*, the same interest rate at which the market discounts the firm's dividend stream to determine its market value. How much would the firm borrow in this situation?

First, assume that the firm can borrow amount $b \ge 0$, how does this affect its investment and dividend policy? Let E(k,b) be the shareholders' equity stake in the firm if it borrows b. This is given by

$$E(k,b) = \max_{0 \le I \le f(k) + b} \left[f(k) + b - I + \beta [V(k(1-\delta) + I) - b/\beta] \right].$$
(20)

Thus, if the firm borrows amount *b* at time t = 0 its total cash for investment is f(k) + b, but the firm must pay back the loan next period, which reduces its equity value by amount $b(1 + r) = b/\beta$ at t = 1. We assume that if the firm's cash flow at t = 1, $f(k(1 - \delta) + I)$ is less than the principal and interest coming due, b/β . If not, there are various ways to deal with the problem. One option is for the firm to borrow via a *consol* which is debt that is repaid via an infinite stream of future debt payments of size $b\beta(1 - \beta)$ in periods t = 1, 2, ... which has present value at time t = 0 of *b*. Another possibility is that some of the firm's shares could be sold to generate the cash necessary to repay the principal and interest to the lender.

Let I(k,b) be the solution to the optimal investment to problem (20). It is easy to see from the previous analysis that if $k > k^*/(1-\delta)$ optimal investment is zero and if the firm borrowed any amount *b* it would just use this to pay current dividends, but in this perfect market scenario, borrowing will not increase the firm's market value. So we assume that the firm will not borrow in this case and we focus instead on the case where $k < k^*/(1-\delta)$. Then it is easy to see that I(k,b) is given by

$$I(k,b) = \begin{cases} f(k) + b & \text{if } f(k) + b < k^* - (1 - \delta)k \\ k^* - (1 - \delta)k & \text{if } f(k) + b \ge k^* - (1 - \delta)k \end{cases}$$
(21)

and this implies an equity value E(k,b) given by

$$E(k,b) = \begin{cases} \beta V(k(1-\delta) + f(k) + b) - b & \text{if } f(k) + b < k^* - (1-\delta)k \\ f(k) - [k^* - (1-\delta)k] + \beta V(k^*) & \text{if } f(k) + b \ge k^* - (1-\delta)k. \end{cases}$$
(22)

Now we define $E(k) = \max_{b\geq 0} E(k,b)$, the firm's maximized equity value when it chooses the optimal level of borrowing. It is easy to see from (22) that the optimal level of borrowing is the function $b^*(k)$ given by

$$b^{*}(k) = \begin{cases} k^{*} - (1 - \delta)k - f(k) & \text{if } f(k) < k^{*} - (1 - \delta)k \\ 0 & \text{otherwise.} \end{cases}$$
(23)

However note that the condition $f(k) < k^* - (1 - \delta)k$ is equivalent to the condition $k < \underline{k}$ where \underline{k} is the lower bound defining the liquidity constrained region in equation (7). Thus we can summarize this by

$$E(k) = \max_{b \ge 0} E(k,b) = \begin{cases} f(k) + \beta V(k^*) - k^* - (1-\delta)k & \text{if } k < \underline{k} \\ V(k) & \text{if } k \ge \underline{k}. \end{cases}$$
(24)

We summarize this result in

Theorem 1 If the firm has access to perfect capital markets with no borrowing constraints, then it will borrow enough in order to reach the optimal steady state capital stock k^* if it is in the liquidity constrained region $k \in [0, \underline{k}]$ but not borrow otherwise. We have $E(k) \ge V(k)$, where V(k) is the value of the firm's equity given in (1) when it cannot borrow, and E(k) > V(k) for $k \in [0, \underline{k}]$. Thus, a sufficiently small, liquidity-constrained firm is strictly better off when it has the option to borrow in a perfect capital market.

Figure 7 illustrates the effect borrowing has on the value of the firm. The gain in value from debt finance decreases rapidly in the size of the initial capital stock k. Clearly the maximum effect occurs at k = 0, since a firm that has no initial capital stock and does not have the ability to borrow cannot get off the ground and has a value of zero. However if firm has access to perfect capital markets, it can borrow the optimal amount $b^*(0) = k^* = 25$ and jump to the optimal steady state capital stock in just a single period, resulting in an equity valuation of E(0) = 50. Note that since we have $\lim_{k \downarrow 0} V(k) = 30.3275$, the gain in value from access to credit markets is smaller, i.e. less than 20, if the firm has even an arbitrarily small amount of initial capital k. This gain from debt financing quickly decreases as the initial capital stock grows, and is negligible once $k \ge 15$. The reason is clear: when the firm has sufficient capital, it produces enough cash flow to finance the bulk of its investment via retained earnings. For example, a firm with initial capital $k_0 = 15$ can reach the optimal steady state capital in just t = 5 periods and has only a single period where it cannot pay dividends to its shareholders. Though borrowing $b = k^* - 15 = 10$ does accelerate the firm's growth and enable it to reach k^* in period 2 and pay higher dividends in periods 1



Figure 7: Effect of borrowing on equity value, $f(k) = \sqrt{k}$

to 5, when we net out the cost of repaying the debt, the gain in value to the initial shareholders is only E(15) - V(15) = 0.0647.

It is also important to consider the *incentive compatibility* of debt financing. That is, would a firm prefer to run off the with loan and consume it rather than investing the loan proceeds as promised to increase the value of the firm? It is easy to see that in the example in figure 7 the firm is better off investing the loan proceeds and paying off the loan than "taking the money and run" and defaulting. For a firm with no initial equity the value of the latter option is $k^* = 25$ whereas the value to the shareholders to investing the loan proceeds in the firm and paying off the loan is E(0) = 50, so the unlimited borrowing limit is incentive compatible in this example.

We also note that the firm has no incentive to borrow in order to pay its shareholders dividends *assuming it pays off the loan*. Clearly, if the firm borrows an amount *b* and pays this in dividends, the present value of future dividends must fall by an equal amount leaving the firm no better off. However borrowing to pay dividends does create an incentive to default if there are no limits on the amount the firm can borrow. If the firm can borrow a sufficiently large amount today and then default, even if the shareholders' equity value is wiped out in the ensuing bankruptcy, the shareholders can be better off. In our subsequent analysis we will impose restrictions on the purpose and the amount the firm can borrow to ensure the incentive compatibility of debt financing, though there is ample evidence from numerous large defaults and firm bankruptcies in the real world that suggest that many actual debt contracts are not incentive compatible.

3.3 Equity financing with perfect capital markets

In the previous section we allowed public firm to finance investment and growth via debt and retained earnings but not via issue of new shares of equity, often referred to as a *seasoned equity offering* (SEO). In this section we will consider the conditions under which existing shareholders might prefer to issue more equity rather than borrow to finance investment. Initially, for simplicity, assume that the firm cannot borrow but the firm can issue new equity with no transactions costs. As we noted in section 2, this is an empirically unrealistic assumption since in reality, there are substantial costs to SEOs and IPOs with both a variable cost of ρ times the amount of equity raised as well as fixed costs *F*. We will subsequently consider the case where ρ and *F* are positive and the firm faces borrowing constraints to provide a richer, more realistic analysis, but at first it is useful to consider what the firm would do if $\rho = 0$ and F = 0.

When a firm issues new shares, we will show the inflow of new capital from the new shareholders acts similar to debt in that it constitutes a form of *leverage* that can increase the value of existing shares. When a firm raises new capital, the existing shareholders much choose how much new equity to raise. Let $\alpha \in (0, 1)$ denote the ownership share of the initial shareholders after the new equity is issued. Existing shareholders also have a choice about how much of the new equity that is raised in the SEO to pay dividends and how much of it to reinvest to finance new investment and growth. Let $\omega \in (0, 1)$ denote the share of the SEO proceeds that the firm pays out to the existing shareholders as dividends.

Let $P(k, \alpha, \omega)$ denote the total proceeds from an SEO by a public firm with initial capital k that issues enough new shares that its post-SEO ownership share is α and it pays out a fraction ω of these proceeds in dividends. We assume that these intentions are common knowledge and indicated in the prospectus available to new shareholders considering buying shares in the company. We derive this function as follows.

First, let *P* denote the net proceeds that result from the SEO that the original shareholders decide to reinvest in the firm. Let E(k, P) denote the equity of the total firm (i.e. both new and existing shareholders) after the management makes a decision about how to optimally allocate these proceeds between investment and dividends. In the absence of debt, we have

$$E(k,P) = \max_{0 \le I \le f(k) + P} [f(k) + P - I + \beta V(k(1-\delta) + I)],$$
(25)

and let I(k, P) be the optimal investment level by the firm when it has capital k and proceeds P. Following the analysis of the previous section where we derived the optimal investment function I(k, b) when the firm had access to debt financing, we have

$$I(k,P) = \begin{cases} f(k) + P & \text{if } f(k) + P < k^* - (1 - \delta)k \\ k^* - (1 - \delta)k & \text{if } f(k) + P \ge k^* - (1 - \delta)k \\ 0 & \text{if } k > k^*/(1 - \delta). \end{cases}$$
(26)

Using the formula for I(k, P) in equation (26) we obtain the following expression for E(k, P)

$$E(k,P) = \begin{cases} \beta V(k(1-\delta) + f(k) + P) & \text{if } f(k) + P < k^* - (1-\delta)k \\ f(k) + P - k^* + (1-\delta)k + \beta V(k^*) & \text{if } f(k) + P \ge k^* - (1-\delta)k \\ V(k) & \text{if } k \ge k^*/(1-\delta). \end{cases}$$
(27)

From equation (27) we see that when the net proceeds from the SEO are sufficiently small (so that the total cash available for investment, P + f(k) is below the incremental investment $k^* - (1 - \delta)k$ necessary to reach the optimal steady state capital k^* , the company invests all cash flow and net SEO proceeds and pays no dividends. However when the cash available for investment exceeds $k^* - (1 - \delta)k$, the firm diverts all of the excess cash, $f(k) + P - [k^* - (1 - \delta)k]$ into paying dividends to *all* shareholders (i.e. both the existing shareholders and the new shareholders who bought a stake in the firm via the SEO).

Let $P(k, \alpha, \omega)$ denote the amount of SEO proceeds raised for a firm with initial capital *k* and where the initial owners choose to retain ownership of a fraction α of the company after the SEO, and use a fraction ω of the net SEO proceeds to cash out (i.e. pay cash to themselves, but not to the new shareholders). Under the assumption of efficient markets and rational expectations, $P(k, \alpha, \omega)$ must be a solution to the following fixed point problem

$$P(k, \alpha, \omega) = (1 - \alpha)E(k, P(k, \alpha, \omega)(1 - \rho)(1 - \omega) - F).$$
(28)

Equation (28) requires the total value of the SEO proceeds on the left hand side equals the expected present value of new investors' equity stake in the post-SEO firm on the right hand side. We assume that the fixed costs *F* for the IPO are paid up front by the existing shareholds. New investors anticipate that the variable costs of the SEO will be deducted from the proceeds, so at most $P(k, \alpha, \omega)(1 - \rho)$ in net SEO proceeds will be available to be reinvested in the firm, given that the fixed costs *F* were already paid before the SEO. If existing shareholders choose to cash out a fraction ω of these net proceeds, then $P(k, \alpha, \omega)(1 - \rho)(1 - \omega) - F$ represents the amount of the original gross SEO proceeds that are actually reinvested in the firm. It follows that the total value of equity after the SEO is $E(k, P(k, \alpha, \omega)(1 - \rho)(1 - \omega) - F)$, and the new shareholders own $(1 - \alpha)$ of this total equity. In equilibrium, the gross SEO proceeds $P(k, \alpha, \omega)$ must equal the value of the post-SEO equity in the firm owned by the new shareholders, which is given on the right hand side of equation (28).

An arbitrage argument can be used to justify equation (28). If the left hand side of (28) is lower than the right hand side, then the SEO is "underpriced" and arbitrageurs can make profits by buying the newly issued shares and then selling them. Conversely, if the left hand side of (28) is larger than the right hand side, then the SEO is "overpriced" and arbitrageurs (or existing shareholders) can profit by selling their existing shares to the new shareholders. Before we consider the optimal level of an SEO (i.e. the optimal choice of α and ω) by existing shareholders, we first establish some properties of $P(k, \alpha, \omega)$ which is an implicitly defined function arising from the fixed point condition (28).

Theorem 2 Consider a public firm with capital stock k and no debt or access to debt financing. Assume that the firm has sufficient capital to pay the up-front fixed cost of the SEO out of current cash flow, F < f(k). Then for any such F, ρ , α and ω not all simultaneously equal to 0, there is a unique fixed point to equation (28) which defines the SEO proceeds as an implicit function of $(k, \alpha, \omega, \rho, F)$ (we suppress the last two parameters to keep the notation compact). The SEO proceeds function $P(k, \alpha, \omega)$ has the following properties:

1. Symmetry Let $V(k, \alpha, \omega)$ be the total market value of the firm after the SEO. We have $V(k, \alpha, \omega) = P(k, \alpha, \omega)/(1 - \alpha)$ for $\alpha \in (0, 1)$. Then $V(k, \alpha, \omega)$ is symmetric in its last two arguments. That is, for any fixed k, ρ and F we have:

$$V(k, \alpha, \omega) = V(k, \omega, \alpha), \quad \forall \alpha \in (0, 1), \ \omega \in (0, 1).$$
(29)

2. Monotonicity $P(k, \alpha, \omega)$ is almost everywhere differentiable in its arguments and satisfies

$$\frac{\partial}{\partial \rho} P(k, \alpha, \omega) < 0 \tag{30}$$

$$\frac{\partial}{\partial \alpha} P(k, \alpha, \omega) < 0 \tag{31}$$

$$\frac{\partial}{\partial \omega} P(k, \alpha, \omega) < 0 \tag{32}$$

$$\frac{\partial}{\partial k} P(k, \alpha, \omega) > 0.$$
(33)

3. Boundary limits For all (α, ω, ρ) sufficiently close to 0, we have

$$P(k, \alpha, \omega) = \frac{(1 - \alpha)E(k)}{1 - (1 - \omega)(1 - \alpha)(1 - \rho)}$$
(34)

where E(k) is the maximal equity value of a firm that in a perfect capital market, equation (24).

The reason we impose the condition that not all of (α, ω, ρ) are zero in Theorem 3 is that no fixed point exists to (28) when k > 0. This can be seen from equation (34) which shows that $P(k, \alpha, \omega)$ tends to $+\infty$ as α, ω and ρ tend to 0. The case where $(\alpha, \omega) = (0, 0)$ is not one we would ordinarily encounter in any case: it would be a situation where the existing shareholders decide to sell out 100% of their interest in the firm, but at the same time invest all of the SEO proceeds back into the company, which would be 100% owned by new shareholders. This is tantamount to the existing shareholders bequeathing their interest in the firm to complete strangers, which is not a situation we would ever expect to see in reality.

To our knowledge, our definition and characterization of the function $P(k, \alpha, \omega)$ for SEO proceeds is a new contribution to the corporate finance literature. Existing theories of SEOs that we are aware of (see,

e.g. Lucas and McDonald [1990] or the real options model of Murray Carlson and Giammarino [2006]) do not have an analog of the fixed point condition (28) determining the value of the SEO proceeds $P(k, \alpha, \omega)$ as a function of the firm's capital k, and the parameters (α, ω) describing the ownership (dilution) and investment decisions made by existing shareholders when they decide to undertake an SEO. The closest antecedent to Theorem 2 that we are aware of is the main theorem of Jensen and Meckling [1976] which states "For a claim on the firm of $(1 - \alpha)$ the outsider will pay only $(1 - \alpha)$ times the value he expects the firm to have, given the induced change in the owner-manager." However their theory is not specific about the "induced change in the owner-manager." However their theory is not specific about the "induced change in the investment in the company following the SEO.

Now we consider a public firm's decision whether to conduct an SEO or not. Besides the discrete decision of whether to do the SEO, the firm also faces a continuous choice over the α (dilution) and ω (cash out) parameters. This problem can be summarized mathematically as

$$\max\left[V(k), \max_{\alpha, \omega} W(k, \alpha, \omega)\right].$$
(35)

where $W(k, \alpha, \omega)$ is the total value of existing shareholders after the SEO, equal to the sum of the anount of the SEO proceeds cashed out plus the value of the existing shareholders' shares in the post-SEO firm given by

$$W(k, \alpha, \omega) = [\omega(1-\rho) + \alpha/(1-\alpha)]P(k, \alpha, \omega).$$
(36)

Thus, the second term in the expression (35) represents the total value to the existing shareholders from conducting an "optimal" SEO.

Theorem 3 *The function* $W(k, \alpha, \omega)$ *has the following properties:*

1. Symmetry If $\rho = 0$, then for all $k \ge 0$ and all $\omega \in (0,1)$ and $\alpha \in (0,1)$ we have

$$W(k, \alpha, \omega) = W(k, \omega, \alpha). \tag{37}$$

2. Monotonicity $W(k, \alpha, \omega)$ is almost everywhere differentiable in its arguments and satisfies

$$\frac{\partial}{\partial \rho} W(k, \alpha, \omega) \leq 0 \tag{38}$$

$$\frac{\partial}{\partial k}W(k,\alpha,\omega) > 0.$$
(39)

3. Boundary limits If $\rho = 0$ and F = 0, then for all (α, ω, ρ) sufficiently close to 0, we have

$$W(k, \alpha, \omega) = E(k), \tag{40}$$

where E(k) is the maximal equity value of a firm that can be achieved in a perfect capital market, given in equation (24).



Figure 8: Value of initial shareholders' wealth for different (α, ω) values

As a consequence of the symmetry of $W(k, \alpha, \omega)$ in its last two arguments, there will always be at least two distinct optimal solutions (ω^*, α^*) and (α^*, ω^*) that maximize the symmetrical formulations of the optimal post-SEO wealth of the existing shareholders. However, result 3 of Theorem 3 implies that in a perfect capital market, *any* pair (α, ω) sufficiently close to the origin maximizes $W(k, \alpha, \omega)$ since this function is flat in this region. This implies that existing shareholders are indifferent between equity and debt financing when capital markets are perfect, and we can frame this as consequence of the *Modigliani Miller Theorem*.

Theorem 4 (Modigliani Miller) If capital markets are perfect and there are no taxes or transactions costs, the existing shareholders of a public firm are indifferent between debt and equity finance, but either is strictly preferred to financing investment via retained earnings if the firm's initial capital stock is sufficiently small, i.e.

$$\max_{\alpha,\omega} W(k,\alpha,\omega) = E(k) = \max_{b \ge 0} E(k,b) \ge V(k), \tag{41}$$

with strict inequality if k is sufficiently small. In particular, E(0) > 0 whereas V(0) = 0.

Figure 8 plots the function $W(k, \alpha, \omega)$ for all possible (α, ω) combinations for a firm with initial capital k = 0 under the assumption that there are zero transactions costs involved in doing a SEO (i.e. $\rho = 0$ and F = 0). As indicated in result 3 of Theorem 3, $W(k, \alpha, \omega)$ is maximized for any combination of (α, ω) that is sufficiently close to zero, and we see this visually in the flat region near the origin in figure 8.

The fact that $W(k, \alpha, \omega) = E(k)$ for so many different combinations of (α, ω) is another manifestion of the Modigliani-Miller Theorem in a world with perfect capital markets. For example W(0, 0, 1/3) = 50, and this corresponds to a decision by existing shareholders to sell out their shares ($\alpha = 0$) but take $\omega = 1/3$

of the SEO as cash diviends, reinvesting the remainder in the firm. The SEO proceeds are P(0,0,1/3) = 150, and so the original shareholders get E(0) = 50 and invest 100 back into the firm. The post-SEO firm invests $I(0,100) = k^* = 25$ of these proceeds and pays out the remaining 75 as cash dividends to the new shareholders. Thus, the value of the post-SEO firm equals the immediate dividend payment of 75 plus the present value of future dividends net of the investment of $k^* = 25$, or equal to $\beta V(k^*) - k^* = 75$, which totals 150, which is the total proceeds from the SEO.

On the other hand, by symmetry we also have W(0, 1/3, 0) = 50. In this case the SEO proceeds are equal to 100, and 100% of these proceeds are invested in the company. After the SEO the company invests $k^* = 25$ in new capital and pays 75 in immediate cash dividends. But this total investment of 75 is a "gift" by the existing shareholders, not incurred by the new shareholders. The total value of the firm equals the sum of the immediate dividend of 75 plus the present value of future dividends equal to $\beta V(k^*) = 75$, or a total value of 150. Since the original shareholders own $\alpha = 1/3$ of the post-SEO company, the value of their shareholdings is W(0, 1/3, 0) = 50 as claimed. In addition, the new shareholders own $(1 - \alpha) = 2/3$ of the post-SEO company, which equals the 100 for the SEO as claimed.

Consider a final case where $\alpha = 0$ and $\omega = 2/3$. Here we can calculate that P(0,0,2/3) = 75, and since the original shareholders sell out and consume $\omega = 2/3$ of the SEO proceeds, their payoff is W(0,0,2/3) =50. The other $(1 - \omega)P(0,0,2/3) = 25$ are invested in the firm and no immediate cash dividends are paid to the post SEO shareholders. However since the *previous* shareholders made the investment of $k^* = 25$, the firm's value to the *new* shareholders (who now own 100% of the firm since $\alpha = 0$) equals $\beta V(k^*) = 75$, and this is exactly the amount of the SEO proceeds.

Recalling the discussion of the case of debt financing in figure 7, we showed that if the firm borrowed $b^*(0) = 25$ it would be able to attain a value of E(0) = 50 despite starting out with no initial capital, which lead to a valuation of zero, V(0) = 0 if the firm could only finance its growth via retained earnings. Thus the firm can attain the same value using debt financing or equity financing, and in the latter case, it can achieve this value with a continuum of different combinations of (α, ω) parameters.

It is easy to extend the definition of SEO proceeds to the case where the firm already has debt. As should be clear from the foregoing discussion, if the firm faces perfect capital markets, there is no additional gain to doing additional equity financing if the firm has already done debt financing or vice versa. Once the firm has reached its optimal steady state capital stock k^* , there is no gain to any additional debt or equity financing since we have $E(k^*) = V(k^*)$ by equation (24) and Theorem 1.

Finally, we discuss the issue of incentive-compatibility of equity financing. Unlike debt financing, where debt holders have the rights to residual control of the company in the case of default, there is an inherent conflict between existing shareholders and new shareholders in an SEO, and the lower the stake of the existing shareholders in the post-SEO firm (i.e. the lower the value of α) the higher the temptation

for the existing shareholders to "take the money and run". For example, in the case where $\alpha = 0$ and $\omega = 1/3$ discussed above, it is not incentive-compatible for the existing shareholders to invest 100 back in the firm. Since they have no stake in the post-SEO firm, they are better off by taking the 150 in SEO proceeds, leaving the new shareholders with nothing.

Or consider the case where $\alpha = 1/3$. This is not a high enough post-SEO ownership stake to give the existing shareholders enough "skin in the game" to avoid the temptation to take the money and run. In the case analyzed above, the SEO proceeds are 100 and the original shareholders promise to invest 100% of these proceeds back into the company, causing its value to rise to 150. However it is clear that the existing shareholders are still better off taking the 100 in SEO proceeds and leaving the new shareholders with nothing and forfeiting their 1/3 stake in the post-SEO company.

However if $\alpha = 2/3$, existing shareholders do have enough "skin in the game." For example P(0, 2/3, 0) = 25, it is incentive-compatible for the existing shareholders to invest 100% of these proceeds back into the firm since this leads to a post-SEO firm value of 75, and the existing shareholders' 2/3 stake in is worth more than consuming the 25 in SEO proceeds as cash dividends.

Thus, in our subsequent analysis, we will impose the following incentive-compatibility constraint on SEO and IPO financing:

$$P(k, \alpha, \omega) + \alpha V(k) \le W(k, \alpha, \omega). \tag{42}$$

The left hand side of inequality (42) is the sum of the SEO proceeds plus the original shareholders' share of the post-SEO company in the event that the initial shareholders renege and do not invest a share ω of the SEO proceeds back in the company but instead consume the entire SEO proceeds as cash dividends. Note that inequality (42) is satisified in the example given above where k = 0, $\alpha = 2/3$ and $\omega = 0$, but is violated in the other two examples where $(k, \alpha, \omega) = (0, 0, 1/3)$ and $(k, \alpha, \omega) = (0, 1/3, 0)$, respectively.

4 Investment and Financial Policy for a Privately Held Firm

In this section we consider the problem faced by a privately owned firm to contrast how its optimal investment and dividend policy differs from that of a public firm. We will show that the behavior of a private firm depends both on the wealth and preferences of the firm's owner (which we assume to be a single individual) as well as the degree of "completeness" of the financial markets in which the firm operates. We start by presenting a "best case" result: *a private owner who has access to complete financial markets will operate his firm exactly the same as a publicly held firm that has access to perfect capital markets.* This is an instance of *Fisher's Separation Theorem* and in this case, there is no reason for a private firm to go public: the smallest transactions cost involved in taking his firm public would induce the private owner to keep his company private. Next we consider deviations from this best case result: 1) a worst case scenario where the owner does not have any financial wealth beside his ownership of the firm, financial markets are incomplete and the owner cannot borrow, and 2) a scenario similar to case 1 except that the owner has access to capital markets but faces binding borrowing constraints.

Consider first the case of a wealthy individual who has financial wealth w and owns the production technology represented by the cash flow production function f(k), though initially we assume that the firm is not yet "founded" in the sense that the owner has not yet purchased any capital in order to get the firm started and producing income. We consider the owner's decision at t = 0 where the owner faces two choices: a) how much to invest in his firm, and b) how much of his remaining wealth to use to purchase an annuity. We assume that financial markets are complete and thus annuities are offered that enable individuals to smooth their consumption streams. Thus, if an individual invests amount w in an annuity at time t = 0, he receives a perpetual annuity payment of $a = w(1 - \beta)$ per period in time periods t = 0, 1, 2, ... where $\beta = 1/(1+r)$ and r is the market interest rate.

Assume that firm's owner has utility function u(c) which is strictly increasing, continuously differentiable and strictly concave. Also assume the owner is a discounted utility maximizer who discounts future utility at rate $\beta_p = 1/(1+r_p)$ where r_p is the owner's personal interest rate, which may or may not equal the market rate of interest r. If this owner devoted all of his wealth to purchasing an annuity, his discounted utility would be $u((1-\beta)w)/(1-\beta_p)$. We now consider whether the owner would be better off by investing some of his wealth in his firm rather than using all of it to buy an annuity.

The owner's problem can be written as follows

$$\max_{\substack{a,k\geq 0\\a+k\leq w}} \left[u(w-k-a) + \frac{\beta_p}{(1-\beta_p)} u(f(k) - \delta k + a(1/\beta - 1)) \right].$$
(43)

Some comments on problem (43) before we present the solution to it. First, we have assumed that the owner is sufficiently wealthy to be able to afford to make a large initial investment of amount k in his firm at time t = 0 but in periods t > 0 the owner will not make additional large investments, but instead only invest enough to cover depreciation of the capital stock δk so that his initial investment will determine a steady state capital stock for the firm equal to the amount of his initial investment k at t = 0. Second, we assume that the owner will choose a level of wealth to annuitize, a, but designates that the annuity payments will start in period t = 1 rather than in period t = 0. This implies that if the owner annuitizes wealth a in period t = 0 he will receive annuity payments of $a(1/\beta - 1)$ in periods t = 1, 2, ... since the present value of this infinite stream of annuity payments at time t = 0 equals the amount annuitized, a.

The first order conditions to problem (43) imply the following solution for (k^*, a^*)

$$k^{*} = f'^{-1}(r+\delta)$$

$$\frac{r}{r_{p}} = \frac{u'(w-k^{*}-a^{*})}{u'(f(k^{*})-\delta k^{*}+ra^{*}).}$$
(44)

Notice that the first equation in (44) is the same as the "golden rule" optimal steady state capital stock for the public firm, given in equation (3) of section 3. Second, notice that if $r = r_p$ then the owner chooses an annuity that results in a constant optimal consumption level c^* in every period equal to

$$c^* = (w - k^*)(1 - \beta) + \beta[f(k^*) - \delta k^*].$$
(45)

Notice this optimal consumption stream equals the value of an annuity purchases from the owner's wealth after investing k^* , $(w - k^*)(1 - \beta)$ plus the stream of dividends that the owner receives from investing k^* in his firm, $f(k^*) - \delta k^*$. Note that these dividends are discounted by β because these dividends start after a one period lag, and are received in periods t = 1, 2, ... and so they must be discounted by β to produce an equivalent annuity stream that is paid out in periods t = 0, 1, 2, ...

The fact that the optimal capital stock that a private owner chooses is the same as the one a public firm chooses when there are perfect capital markets is striking: the owner's preferences (β_p , u) have *no impact* on his production decisions when financial markets are perfect and complete. The owner's preferences only affect how he annuitizes, in order to best smooth the resulting dividend stream and his initial endowment of wealth at t = 0. As we noted above, if $r = r_p$, then the owner chooses to have a flat consumption profile given by equation (45).

In our solution (44) we assumed that the private owner's initial endowment of wealth is sufficiently large that $w \ge k^*$. However if $w < k^*$ the owner can use financial markets to finance any shortfall and reach the desired capital stock k^* via a consol with an infinite stream of interest payments equal to rb where b is the amount borrowed in period t = 0, which is paid off in a level stream of repayments starting in period t = 1 and continuing in perpetuity. It is easy to see that if unlimited borrowing is allowed, the owner can achieve the same solution, so the optimal solution is the same as given above, but with $b^* = -a^*$.

Theorem 5 (Fisher's Separation Theorem) Suppose financial markets are perfect and complete. An owner of a private firm will choose the same production policy as the owner of a public firm, independent of his preferences. Preferences only affect how the private owner chooses to smooth his consumption stream using financial markets.

Theorem 5 can be viewed as an analog of the Modigliani-Miller Theorem. The latter theorem states that when financial markets are perfect and there are no taxes or transactions costs, then *capital structure is irrelevant*. That is, the firm's investment policy and the total value of a *public firm* is independent of how it is financed. Theorem 5 tells us that under the same conditions, *ownership structure is irrelevant*. That is, a private and public firm will make exactly the same investment decisions, regardless of the preferences of a private owner. We call Theorem 5 "Fisher's Separation Theorem" since it is a special case of a general principle that Irving Fisher [1930] noted in *The Theory of Interest*, namely the owner of a private company "has, therefore, two kinds of choice: first, the choosing one from many optional income streams, and secondly, as under the first approximation, the choosing of the most desirable time shape of his income

stream by exchanging present income against future." (II.VI.2). Fisher argued that the owner's utility maximization problem can be decomposed into two subproblems: 1) choose an investment strategy that maximizes the expected present value of dividend streams, and 2) use financial instruments to choose the most desirable time shape of his consumption stream, subject to the constraint that the expected present value of his dividend the expected present value of his dividend (income) stream.

As we noted above, when financial markets are complete and frictionless, there is no gain to the owner of a private firm in taking his company public. The owner can simultaneously maximize and smooth his consumption and obtain the same consumption stream he could obtain by taking his company public and selling his shareholdings to purchase an annuity. Thus, the slightest transaction cost involved taking a private firm public would induce the owner of a private firm to stay private. We now relax the complete, frictionless financial market assumption that Fisher's Separation Theorem to better understand the conditions under which a private firm might want to go public, even in the presence of transactions costs.

Now consider the opposite extreme: assume the owner cannot borrow and does not have access to annuities. Assume the owner has invested all of his initial wealth w to buy an initial capital stock k. Let W(k) represent the present discounted utility of an owner of a private firm with capital stock k. The Bellman equation for the privately held firm is given by

$$W(k) = \max_{0 \le I \le f(k)} \left[u(f(k) - I) + \beta_p W(k(1 - \delta) + I) \right].$$
(46)

Note that we use the notation W(k) instead of V(k) since W(k) represents the welfare of the private owner who has initial capital k and is measured in utility units, whereas V(k) represents the value of a public company with capital k and is measured in dollars. The first order condition for optimal investment is given by

$$u'(f(k) - I(k)) = \beta_p W'(k(1 - \delta) + I(k)).$$
(47)

From the "Inada condition" i.e. that $\lim_{c\downarrow 0} u'(c) = +\infty$, it is easy to see that the optimal investment policy will always entail paying some positive level of dividends, i.e. I(k) < f(k) for all k. However it may still be the case that if the firm had sufficient capital, it may be optimal not to invest, i.e. I(k) = 0 for $k \ge \overline{k}$, though the value of \overline{k} may be different than the value $\overline{k} = k^*/(1-\delta)$ at which a public firm stops investing. Using the Envelope theorem, we have

$$W'(k) = u'(f(k) - I(k))f'(k) + \beta_p W'((1 - \beta)k + I(k))(1 - \delta),$$
(48)

but using the first order condition (47) we have

$$W'(k) = u'(f(k) - I(k))[f'(k) + (1 - \delta)],$$
(49)

and substituting this back into the first order condition (47) we can derive the Euler equation characterizing

the private investor's optimal investment policy I(k)

$$u'(f(k) - I(k)) = \beta_p u'(f(k(1 - \delta) + I(k)) - I(k(1 - \delta) + I(k))) \left[f'(k(1 - \delta) + I(k)) + (1 - \delta)\right].$$
 (50)

This is a non-linear functional equation for I and it is ordinarily not an easy one to solve via numerical methods. It is not clear there there is a closed form solution in this case, unlike the one we found for the optimal investment policy of a publicly held firm.

However in steady state we have $I(k^*) = \delta k^*$ and substituting this for I(k) in the Euler equation above we obtain

$$u'(f(k) - \delta k) = \beta_p u'(f(k) - \delta k)) \left[f'(k) + (1 - \delta) \right],$$
(51)

or $f'(k) = 1/\beta_p - 1 + \delta = r_p + \delta$, for which the only solution is $k = k_p^*$. Similar to the case of a public firm, if a private owner does not have sufficient initial wealth to invest in the firm at the optimal level k_p^* and cannot borrow, he can finance growth via retained earnings and the firm will gradually accumulate capital and converge to the optimal steady state k_p^* asymptotically.

Note that if $\beta_p = \beta$ (and thus $r = r_p$) then $k_p^* = k^*$: the steady state capital stock for a privately owned firm is the same as a public firm. However if $r_p > r$, then it is easy to see from the Golden rule condition (4) that $k_p^* < k^*$, and vice versa. That is, when the private owner is more impatient than the the market as a whole, the private firm will have a smaller scale and value in steady state than a public firm. Thus, there are two key differences between the behavior of a private and public firm when financial markets are incomplete: 1) the owner of a private firm always pays positive dividends, and accepts slower growth as a necessary price for "dividend smoothing", and 2) if $r \neq r_p$, the steady state size of the firm differs from the steady state size of a public firm.

Figure 9 plots the optimal investment and dividend policy functions for a privately held firm where $r = r_p = 0.05$ and compares them to the ones chosen by a publicly held firm.⁷ We see that except for the steady state, the investment and financial policy of a private and public firm are quite different from each other. The top left panel shows the optimal investment policies for the two firms plus the level of replacement investment necessary to keep the capital stock from declining. The intersection of the optimal investment curves and the black replacement investment line defines the optimal steady state capital stock level k^* and as predicted by our analysis above, we see that it is the same for both the public and privately held firm.

Away from the steady state, investment and dividends are quite different from each other. Investment by the privately held firm is less than investment by the public firm for $k \in (0, k^*]$, but investment by the privately held firm is greater than investment by the public firm for $k > k^*$. The pattern for dividends is the

⁷The solutions for the privately held firm were calculated numerically using the discrete policy iteration algorithm described in the appendix.



Figure 9: Optimal investment and dividend policy and value function for $f(k) = \sqrt{k}$

opposite: the private firm pays higher dividends than the public firm for $k \in (0, k^*]$, but lower dividends for $k > k^*$, unless capital is sufficiently high that both the public and private firm stop investing, and in this region the dividend payments coincide. The dividend functions also intersect at the steady state capital stock k^* , so that asymptotically as $t \to \infty$ the behavior (i.e. investment and dividends) of the two firms coincide.

The lower left panel of figure 9 plots the value of the privately held firm V(k) and compares it to the utility the investor would have obtained if they invested all of their wealth in an annuity earning the market rate of return. We see that at least if investment is framed as an all or nothing choice, it is always preferable for the investor to invest their wealth in the private firm rather than in an annuity. Investing in their own firm generates much higher returns, dominate the r = .05 return that the person can obtain from an annuity. Another way to see this is to look at the black line in the right hand top panel of figure 9. This plots the annuity income the investor would receive each period if they invested all of their wealth into an annuity. We see that the dividend income from investing in a private firm dominates the annuity income they would receive at all levels of initial investment k.

Finally, the lower right hand panel of figure 9 compares the evolution of investment and capital stock for a public and a private firm that each begin life with an initial capital stock of k = 1. We see that due





to the higher early investment, the public firm reaches the steady state capital stock $k^* = 25$ after only 15 periods, whereas the privately held firm approaches k^* only asymptotically. We summarize our analysis of the private firm in Theorem 6 below.

Theorem 6 (behavior of a private firm in imperfect financial markets) Consider a privately owned firm where the owner has a concave utility function u(c) and production function f(k) where u satisfies the Inada condition. Assume the firm cannot borrow and does not have the option to do an IPO in the future. Then the optimal dividend policy is to pay positive dividends for any positive level of the capital stock. The privately owned firm adopts an inefficient investment policy (i.e. a policy that does not maximize its market valuation) due to the owner's desire to use dividends to consumption-smooth, resulting in a slower rate of accumulation of capital. The steady state capital stock of a privately owned firm, k_p^* , is the same as a publicly owned firm if and only if $r = r_p$, i.e. the private owner's rate of discount r_p is the same as the market discount rate, r.

We conclude this section by illustrating the non-concave case, where the owner has a concave utility function but the production technology f(k) is non-concave for the reasons explained in section 3. Figure 10 illustrates the optimal investment and dividend policy for the owner of a private firm who faces the same non-concave production technology that we solved for a public firm in section 3 (see figure 6). Similar to the concave case analyzed above, the owner of a private firm distorts his investment in order to generate positive dividends in all circumstances. We also see that the private firm has the same two Golden Rule steady state values of k^* that we computed for the public firm, but with different domains of attraction. The owner of a private firm needs substantially more initial capital, more than k = 125, to be willing to invest and grow to reach the higher steady state value of $k^* = 222$, whereas a public firm only needs k in excess of 80 to reach this higher Golden Rule capital stock k^* .

We can see from the right hand panel of figure 10 that when k is around 150 the private owner is willing

to get by on very little dividend income in order to rapidly invest and reach the higher steady state k^* in order to enjoy a permanent dividend stream of $D(k^*) = 18.12$ net of depreciation expenses of $\delta k^* = 11$. However the main message is that similar to the case of a public firm, borrowing constraints can create "liquidity traps" that cause firms to forgo attractive investment opportunities and remain a small. Though the private owner is behaving optimally given his financial constraints, the behavior is suboptimal relative to a world where the owner could borrow enough to reach the higher steady state k^* right away, instead of facing the delay of having to slowly build up enough capital by financing investment via retained earnings. In the next section we will study how borrowing can help the firm avoid this globally suboptimal outcome.

5 Modeling the IPO Decision

We now extend our model of a privately owned firm to give the owner the option of "taking the firm public" via an IPO. Once the firm is public, we assume it is run by a manager who maximizes the present value of dividends. By selling off his 100% stake in the firm, the owner no longer has any operating control, but he can take the proceeds raised by the IPO to start a new company, buy an annuity, or invest the proceeds in financial securities and live happily ever after on the interest and principal income. What will the owner decide to do: sell his firm in an IPO, or keep it private?

We start by considering the simplest case where neither the private or public firm have the option to borrow. This is a relevant point of departure since by assuming the neither firm can borrow we remove any motive for the private owner to go public in order for gain access to credit markets (which is a typically cited reason for going public as discussed in section 2). Thus, after selling out the newly created public firm will have the same level of capital and will continue to have to rely on retained earnings to finance investment. We show that if the costs of undertaking an IPO are not too high, the private owner owner will be better off by selling out rather than continuing to operate his firm privately.

To see why, note from section 4 that the owner of a private firm adopts an *inefficient* investment policy as a result of his desire to consumption smooth. This leads to the owner to "dividend smooth" which comes at a cost in terms of firm growth and the market value of the company. We can quantify this inefficiency in terms of forgone market value. Consider the case where neither a public or private firm can borrow. Let $I_p(k)$ be the optimal investment policy adopted by the owner of a private firm, and let $D_p(k) = f(k) - I_p(k)$ be the corresponding dividend, from the solution to the private owner's problem in the Bellman equation (46). Define $V_p(k)$ as the market valuation of a firm that adopts the investment and dividend policy of a private firm, i.e. (D_p, I_p) . V_p is the unique solution to the equation

$$V_p(k) = D_p(k) + \beta V_p(k(1 - \delta) + I_p(k)).$$
(52)

Let (D,I) be the dividend and investment policy from the solution to the public firm's problem in the



Figure 11: Ratio of $V(k)/V_p(k)$ for $r_p = 0.05$ and $r_p = 0.07$ and $f(k) = \sqrt{k}$

absence of access to capital markets, the Bellman equation (1), and let V(k) be the corresponding value function. Since (D,I) is an optimal policy and (D_p,I_p) isn't, we have the following result

Theorem 7 (Inefficiency of a private firm in incomplete financial markets) Suppose that neither public nor private firms haave access to capital markets. The optimal financial and investment policy of the private firm is inefficient in the sense that

$$V(k) \ge V_p(k) \tag{53}$$

except in the case where $r = r_p$, then $k^* = k_p^*$ and $V(k^*) = V_p(k_p^*)$.

Figure 11 illustrates the magnitude of the inefficiency caused by private ownership by plotting the ratio $V(k)/V_p(k)$ under two different scenarios: 1) r = 0.05 and 2) r = 0.07. We see that there is a large proportional increase in the value of the firm from going public and this gain is the largest for smaller firms (i.e. smaller values of k). We also see that the proportional gain in value is larger when $r_p \neq r$, since in that case we have $k^* \neq k_p^*$ and there is an additional inefficiency since the private firm's steady state scale of operations differs from the efficient (profit maximizing) scale of operation k^* that a public firm would choose. In the case where $r_p = 0.07$, we have $k_p^* = 17.36$ which is 30% lower than the efficient steady state capital stock $k^* = 25$ that a public firm converges to in steady state.

Thus, Theorems 6 and 7 tell us that Fisher's Separation Theorem fails when there are financial frictions, the most extreme case being when the firm does not have access to capital markets. The cost of this inefficiency can be quantified via the loss in market value $V_p(k) - V(k)$ that the private owner voluntarily incurs by distorting his investment and dividend policy to smooth his consumption stream. However this creates the motive for going public, which can be viewed as a weaker form of Fisher's Separation Theorem:

by going public a private owner achieves a separation of ownership and control of the firm, which enables it to avoid the inefficiency of private ownership and enables the owner to capture the gain in value and smooth consumption more efficiently in financial markets.

If it is costless to go public and owners do not receive non-pecuniary value/utility from controlling and running their own private firm, then it is clear from figure 11 that all firms should go public and operate as risk-neutral expected discounted dividend maximizers. However when there are costs to going public, things are more complicated. Assume that the fixed costs of an IPO must be paid up front, prior to the IPO. Define \underline{k} by $f(\underline{k}) = F$. We will also show that when ρ or F are non-zero, there will be another threshold $\overline{k} > \underline{k}$ such that for $k > \overline{k}$ it is too expensive to go public, and the owner will choose to remain a private firm.

Theorem 8 (Conditions for going public) Consider a firm that is initially privately held by an owner whose personal subjective discount rate r_p may differ from the market interest rate r at which dividends of publicly traded firms are discounted at. If the owner can save at interest rate r or if there is an actuarially fair annuity market which also earns the market return r, then if there are no costs to an IPO, the owner will always prefer to go public and use the proceeds to save and consume or purchase an annuity, except if k = 0 or if $r = r_p$ and $k = k^*$, where the owner is indifferent about cashing out or not. If there is a proportional fee $\rho \in (0,1)$ in an IPO, (so the owner only receives the share $(1 - \rho)$ of the IPO proceeds), then only private firms with sufficiently low levels of initial capital k will find it optimal to do an IPO. If there are also up-front fixed costs F required to do an IPO, then if ρ and F are not too high, there will be an interval of values of the private owner's capital stock, $[\underline{k}, \overline{k}]$ where it is optimal for the firm to go public via an IPO. An owner with capital less than \underline{k} will not have sufficient size to afford the fixed costs of undertaking an IPO, and an owner with capital greater than \overline{k} can afford to do the IPO but finds the transactions costs too high to make it worthwhile.

In the case of no transactions costs to doing an IPO, the proof of Theorem 8 is intuitively clear: we showed that the private owner has a motive to consumption smooth, but this motive distorts the owner's investment policy, since his desire to pay dividends in every period slows the rate of accumulation of capital to the optimal steady state value k^* . By going public, the newly public firm will avoid these inefficiencies and the owner can continue to consumption-smooth by using the IPO proceeds to purchase an annuity. This result is, in effect, a type of *separation theorem* between investment and consumption. It shows that in the absence of transactions costs, firms ought to be publicly held rather than privately held, since it is more efficient to use capital markets than investment policy to smooth out consumption streams. However if k = 0, then since neither the public or private firm can borrow the owner gets zero in either case and cannot gain from doing an IPO. If $k = k^*$, then the firm is already at the optimal steady state capital stock, and since this results in a flat consumption stream of $f(k^*) - \delta k^*$ per period (the same as what an annuity

would pay the owner if he sold out), there is no gain from doing an IPO in this case either.

Recall that W(k) represented the welfare (discounted utility) of a private owner who did not have the option to go public, and did not have access to credit markets. Let $W_{ipo}(k)$ be discounted utility of a private owner who exercises the option to go public, under the assumption that the owner does a 100% sell out of his ownership interest. We have

$$W_{ipo}(k) = \frac{u((1-\beta)[V(k)(1-\rho)-F])}{(1-\beta_p)}.$$
(54)

 $W_{ipo}(k)$ represents the value of "stopping" for a private owner, i.e. the value of the option to go public. It is the discounted utility of the value of an annuity that pays an amount $a = (1 - \beta)[V(k)(1 - \rho) - F]$ in perpetuity that the owner can receive after selling his firm and receiving its market valuation V(k) net of the proportional and fixed underwriting fees. Let W(k) be the discounted utility for the owner of a private firm who has the option of going public in the future. Then we have

$$W(k) = \begin{cases} \max_{0 \le I \le f(k)} \left[u(f(k) - I) + \beta_p W(k(1 - \delta) + I) \right] & k < \underline{k} \\ \max \left[W_{ipo}(k), \max_{0 \le I \le f(k)} \left[u(f(k) - I) + \beta_p W(k(1 - \delta) + I) \right] \right] & k \ge \underline{k} \end{cases}$$
(55)

This equation is similar to the Bellman equation (46) for the owner of a private firm without the IPO option that we presented in section 4, except that now problem (55) takes the form of an *optimal stopping problem*. The solution is defined by a partition of the state space into a *continuation region* where it is optimal for the firm to continue to stay private and a *stopping region* where it is optimal for the firm does not generate enough cash flow to afford the up-front fixed costs to undertake an IPO, so the owner's only option is to stay private in this interval with a value given by the top line of equation (55). However for $k \ge k$ the owner can afford to undertake an IPO, but his decision is based on whether net of the proportional costs of the IPO (i.e. net of the fraction ρ deducted from the sales proceeds V(k) in equation (54)) he will be better off selling out or staying private.

Then the private owner cannot afford to go public if his initial capital stock is in the interval $[0,\underline{k})$, so this is part of the continuation region. We can also show that if ρ is not too large, there will be a unique value \overline{k} that satisfies $W_{ipo}(\overline{k}) = W(\overline{k})$ and $W_{ipo}(k) < W(k)$ for $k < \overline{k}$ and $W_{ipo}(k) > W(k)$ for $k > \overline{k}$. Thus the stopping region is the interval $[\underline{k},\overline{k}]$ and the continuation region consists of the two intervals $[0,\underline{k})$ and (\overline{k},∞) .

The left hand panel of figure 12 illustrates the gains to going public for a private owner with utility function $u(c) = c^{.7}$ where $r_m = r_p = 0.05$ and the proportional transactions costs for doing an IPO is $\rho = 0$. We see that consistent with Theorem 8, the gains from an IPO are positive for all values of *k* except k = 0 and $k = k^* = 25$ where they are zero. The right hand panel of figure 12 illustrates the gains to going public when $\rho = 0.07$ and the fixed fee F = 0.5. The 7 percent commission rate is typical for IPOs underwritten

Figure 12: Effect of transactions costs ρ on the value of going public



by investment banks in the US and other countries. We see with the higher commission rate, the gains from doing an IPO are negative for $k > \overline{k} = 2.75$, and also an owner who has no access to credit markets is unable to afford the fixed cost of the IPO when $k < \underline{k} = .25$ since the cash flow $f(k) = \sqrt{k}$ is insufficient to enable the owner to pay *F* in this case. So the interval $[\underline{k}, \overline{k}] = (.25, 2.75)$ represents the optimal stopping region in this case: the interval of capital where it is optimal for the owner to take his firm public with an IPO and then sell out.

We see that in the presence of transactions costs, it is no longer better to go public regardless of the initial capital stock of the firm. It is only optimal for firms that are in the "goldilocks zone" with a minimal amount of capital to be able to afford the fixed costs of an IPO but not too much capital where the commission constitutes an unacceptably large tax on the owner. In addition, as the private firm's capital gets close to k^* , the incentive to do an IPO falls as we showed in Theorem 8, since the firm is close to a point where it can reach k^* quickly anyway using its own retained earnings. Thus, we have succeeded already in providing an answer to both "why" and "when" private firms go public.

Now we consider an additional motivation for a firm to go public: access to credit markets. It is often claimed that due to agency reasons and information costs, public firms have superior access to credit markets than private firms. We will start by considering the most extreme case where the owner has no ability to borrow as a private firm but if the firm were public, it would face no borrowing constraints. Clearly, when this is the case the incentive to go public is enhanced, since we have already shown how the ability to borrow increases the value of a public company.

Corollary 8.1 (Access to credit markets enhances the value of going public) Suppose the assumptions of Theorem 8 hold except that public firms have unlimited access to credit markets and face no borrowing constraints at an interest rate $r_b = r_m$. Then the incentives for going public are higher, and when there are both fixed and proportional transactions costs to undertaking an IPO, the interval of capital $[\underline{k}, \overline{k}]$ where

Figure 13: Gains to going public with $\rho = 0.07$ when post-IPO public firm has access to credit markets



it is optimal for the owner to do an IPO is larger than in the case where public firms have no access to capital markets.

Figure 13 illustrates Corollary 8.1 by plotting the gains to going public, $W_{ipo}(k) - W(k)$, in the case where the post-IPO public firm has full access to credit markets and can borrow unlimited amounts at $r_b = 0.05$, the same as the owner's personal interest rate and the rate at which the public firm's dividends are discounted. The biggest gain to going public now occurs at k = 0: an owner with no initial capital cannot borrow as a private firm and thus cannot get his firm off the ground. But once the firm goes public it obtains access to credit markets, which enables the firm to borrow enough to attain the optimal steady state capital stock k^* with only a single period delay. Thus, the owner of a private firm is able to unlock tremendous value by the act of going public, and in effect, gain access to credit in an indirect manner that would not be possible for him to do if the firm remained private.

We can see from figure 13 that due to the increased value that access to credit markets confers on the post-IPO public firm, the interval of capital $[\underline{k}, \overline{k}]$ for which going public is optimal widens from [.25, 2.75] to [.25, 5]. We can see that the gains to doing an IPO decline very rapidly as k increases, and private firms with the least capital are the ones that gain the most from going public. Underwriters of IPOs might take note of this and waive any up-front fixed fees for doing an IPO for the smallest private firms in exchange for a larger commission ρ . For example if the underwriter waived the fixed fee, F = 0 and raised the commission from $\rho = 0.07$ to $\rho = 0.08$, it would capture new business from the smallest private firms and the incremental commission revenue (since the valuation of a new public firm is V(0) = 50), makes up for lost fixed fee. However fewer medium-sized private firms would do IPOs, since \overline{k} falls from 5 when $\rho = .07$ to $\overline{k} = 4.25$ when $\rho = 0.08$.

Finally, suppose both private and public firms have equal (unlimited) access to credit markets at an

interest rate $r_b = r_m = r_p$. Then the gain to doing an IPO drops precipitously, and unless the transactions costs of doing an IPO are zero, no private firms will choose to go public. This follows because with unlimited ability to borrow, as we will showed in Fisher's Separation Theorem (Theorem 4 of section 4), the private owner is able to reach the efficient steady state k^* in the first period, after which the private owner achieves the same dividend stream in perpetuity as a public firm produces.

Corollary 8.2 (Access to credit markets by a private held firm reduces the value of going public) *If the owner of the privately held firm has access to credit markets, the incentives for undertaking an IPO are dramatically reduced, and unless the transactions costs associated with undertaking an IPO are near zero, the private owner will have no incentive to go public. The incentives for going public are stronger if: a) the borrowing constraints the private owner faces are tighter, and b) the owner of the private firm is encumbered with initial debt.*

Results a) and b) of Corollary 8.2 are intuitively clear, but order to formally prove that they hold we need to extend the private owner's investment and financing problem to account for debt financing and credit constraints, and the possibility that in order to start the firm the founder may have to go into debt to finance the acquistion of the production technology f and initial capital stock k, but due to borrowing constraints, the owner cannot borrow enough to reach the efficient steady state capital stock k^* imediately. We will consider this extension of the private owner's problem in section 6 and show how too much initial debt can encumber the private firm and increase the incentives for doing an IPO.

5.1 Using IPOs to raise new capital

As we noted in section 2 most IPOs do not entail a 100% sell-off of the original owner's stake in the company. Instead, the original owner retains a partial ownership stake in the firm, and only takes part of the IPO proceeds in cash to finance consumption or other investment projects. The other important role of a partial cash-out is that when the original owner continues to own a significant share of the post-IPO company, the share of the IPO proceeds that the owner does not cash out can be re-invested in the company, thereby providing a new infusion of capital to the firm after the IPO that is not reflected in our analysis of an IPO with a 100% cash out by the original owner.

The cash raised in an IPO thus has similarities to the proceeds of a loan: in each case the original owners have "sold off" some of the future dividends of the company in order to finance current investment. When the returns from investment are sufficiently high, the increased value from the investment in the firm outweighs the future claims of debt-holders or new shareholders, increasing the overall equity value of the original owner compared to a scenario of a 100% sell off that we considered in section 5. In this section we characterize the conditions under which the original owner will want to do an IPO but retain only a partial ownership stake in the post-IPO company and reinvest some or all of the capital raised in the IPO.

Thus, an IPO benefits the owner in two different ways: 1) the IPO changes the objective function of the firm from utility maximization to discounted value maximization, benefitting the original owner by eliminating what we might call the *dividend smoothing inefficiency* of private ownership, and 2) the IPO provides an infusion of new capital that the original owner can use to reinvest back into the company.

Suppose the firm is originally a privately owned firm by a sole owner, and the owner chooses to take the firm public via an IPO and retain only a fraction $\alpha \in (0, 1)$ of his/her original 100% ownership stake in the firm. Thus, after the IPO the original owner will own a fraction α of the firm (i.e. α is fraction of shareholdings still owned by the founder of the firm) and the outside investors who bought shares in the new firm will own the remaining fraction $1 - \alpha$ of the firm's shares.

The IPO proceeds equal the total amount the founder receives from selling shares in the newly public firm to the new outside investors, net of the underwriting fees. Mathematically, at least in terms of our formulation of a fixed point condition that determines the proceeds raised, an IPO is equivalent to the case of a SEO that we introduced in equation (28) of section 3.3. In particular, the founder can either reinvest the net proceeds to increase the capital stock (and hence future profit/dividend stream of the firm), or cash out some or all of the proceeds for private consumption or investment purposes (e.g. to buy an annuity or invest in another venture). As in the case of an SEO we let $\omega \in [0, 1]$ represent the fraction of the IPO proceeds that the owner chooses to take out for consumption or other investment purposes, and thus the fraction $1 - \omega$ of the net IPO proceeds is reinvested in the firm.

There is, of course, a key practical difference between a SEO and an IPO. In a SEO there is much more of a "track record" and more information available to potential new shareholders about the company and its prospects. In an IPO, there is less information about the new company given that it has been run as private operation until the point where the IPO takes place. Information about the prospective new company is provided in a prospectus prepared by an underwriter as one of their key services in conducting the IPO.⁸ Part of the concern and risk to new shareholders considering investing in an IPO is the possibility that the founder might "take the money and run" after the IPO. One of the key functions provided by an underwriter is to do the *due diligence* to investigate the private firm that wishes to go public with an IPO and certify to investors that the company really does exist and the founder will not "take the money and run" after an IPO. Thus, the reputation of the underwriter, in addition to market regulation (such as is done by government agencies such as the Securities and Exchange Commission), helps convince outside investors that an IPO is legitimate and is not a thinly disguised take the money and run scheme.

However we believe that as a first approximation that information on the firm's production technology f, its initial capital k, and the founder's choice of (α, ω) can be treated as credible common knowledge to

⁸As Draho [2004] notes, "The prospectus is usually the first source of information about the issuer available to investors, as their is rarely anything published prior to the IPO (Rao 1993). The prospectus provides a thorough overview of the firm, its operations and structure, intended use of proceeds and terms of the offering." (p. 164).



Figure 14: Value of founder's post-IPO wealth for different (α, ω) values

investors that is revealed by the due diligence of the underwriter and communicated in the prospectus for the IPO. Of course the underwriter incurs costs of doing the due diligence and preparing the prospectus, and advertising the new issue to attract a sufficient number of new investors, and providing liquidity and avoid adverse price impacts by selling too many shares too quickly. that a private firm that wants to go public via an IPO is legitimate. Some underwriters provide insurance to the founder via a firm-commitment contract that fully or partially guarantees some level of proceeds from the IPO as discussed in section 2.2. The underwriter is compensated for this risk and its services by charging a proportional fee $\rho \in (0,1)$ plus a fixed fee *F*. Thus if the gross proceeds of the IPO are $P(k, \alpha, \omega)$, the net proceeds received by the founder from the investment bank (after it deducts its fees) are $(1 - \rho)P(k, \alpha, \omega) - F$.

Figure 14 plots the value of the founder's post-IPO wealth (the sum of the amount cashed out from the IPO plus the value of the shareholding in the post-IPO company that the founder retains) as a function of (α, ω) for $\underline{k} = .25$, the smallest level of *k* that enables the firm to afford the fixed costs of the IPO: f(.25) = .5. In equation (36) of section 3.3 we defined the function $W(k, \alpha, \omega)$ to represent the total wealth of the original shareholders after an SEO, and this same function also characterizes the founder's total wealth after an IPO. Although it is difficult to see from figure 14, when $\rho > 0$ and F > 0, the function $W(k, \alpha, \omega)$ is no longer symmetric in its (α, ω) arguments. For example, $W(.25, .7, .2) = 48.68 \neq W(.25, .2, .7) = 46.43$.

Figure 15 provides further on the impact of the α and ω parameters detail by plotting slices of the function $W(k, \alpha, \omega)$. The left hand panel of figure 15 shows how the value of the firm varies with α when we fix $\omega = .2$. Of course, the IPO proceeds must equal 0 when $\alpha = 1$ and the value of the post-IPO firm increases monotonically as α decreases. This is due to the fact that the founder is reinvesting $\omega = .2$ of the



Figure 15: Value of founder's post-IPO wealth as a function of α and ω

IPO proceeds back in the firm and thus the larger the fraction of firm the founder sells off (i.e. the smaller is α), the greater the amount that is reinvested, thereby increasing the value of the firm. The solid black line plots the net IPO proceeds and the dashed line plots the share of the net proceeds that are reinvested (the remaining proceeds are used to pay dividends). In this case, the value of the founder's total wealth (cash taken out plus ownership in the post-IPO firm) is maximized when $\alpha = .739$, resulting in total wealth of W(.25, .739, .2) = 48.73.

The right hand panel of figure 15 shows how the founder's wealth varies with ω when α is fixed at $\alpha = .5$. Of course, the total post-IPO equity value (red line in the figure) is maximized when $\omega = 0$ since the founder has reinvested all of the net IPO proceeds back into the company. However this does not maximize the founder's wealth since the outside investors are receiving half of the benefits of the founder's investment of the IPO proceeds. The founder's total wealth is maximized by setting $\omega = .4686$, resulting in total wealth of W(.25, .5, .4686) = 47.80. By cashing out this relatively large share of the net IPO proceeds the founder leaves the company with insufficient funds to reach the optimal steady state capital stock k^* . In comparison, when $\omega = 0$, the net proceeds are W(.25, .5, .0) = 43.16, which exceeds the capital the firm needs to invest to reach $k^* = 25$. So the firm pays out difference, 43.16 - 25 = 18.16, as cash dividends and the founder receives half of this amount.

When we simultaneously optimize over (α, ω) which combination maximizes $W(k, \alpha, \omega)$? Intuitively, it should not be optimal for the founder to use any of the net IPO proceeds for cashing out if the return from investing these proceeds has a sufficiently high rate of return. Of course, there is a "tax" of ρ imposed by the underwriter on each dollar raised in the IPO. Cashing out these "after tax" dollars has a marginal return of 1 (i.e. allocating one dollar of the net IPO proceeds to cashing out increases the founder's wealth by one dollar). However reinvesting the marginal dollar back into the firm has a marginal return of $\alpha\beta V'(k + P(k, \alpha, \omega)(1 - \rho) - F)$. As long as this latter rate of return is greater than 1, the founder is better off reinvesting the marginal dollar from the net IPO proceeds rather than cashing out. This will be the case if after receiving the net IPO proceeds, the firm is still in the "investment region" in equation (27) where it optimally allocates all resources to investment and pays no dividends. This will be the case when the initial capital stock k is sufficiently low and α is sufficiently high, i.e. when the founder has retained a sufficiently large ownership share of the post-IPO firm. We summarize this login in Theorem 9 below.

Theorem 9 If $\rho > 0$ then $W(k, \alpha, \omega)$ is not symmetric in (ω, α) . For sufficiently large $\alpha W(k, \alpha, \omega)$ is decreasing in ω . Thus, the value of (α^*, ω^*) that maximizes $W(k, \alpha, \omega)$ satisfies $\omega^* = 0$, i.e. if a wealth-maximizing founder chooses to take his company public, he will choose to reinvest all of the net IPO proceeds back into the company.

To illustrate Theorem 9, consider a founder who has initial capital, k = 0.25 which is just sufficient to pay for the fixed cost of an IPO, F = 0.5. If the owner were to sell 100% of his firm and not reinvest any of the proceeds (i.e. $\alpha = 1$ and $\omega = 1$) the owner's wealth would be W(.25, 1, 1) = 42.68, which equals $\beta V(k(1 - \delta))$ since the fixed cost of the IPO consumes the firm's entire cash flow, so it can neither invest nor pay dividends in the period it goes public. The the optimal IPO occurs at the parameters (α^*, ω^*) = (.776,0) which yields the owner a post-IPO total wealth of W(.25,.776,0) = 48.96. The founder is better off from doing the IPO, since even net of the fixed fee for doing the IPO, F = 0.5, his wealth is higher from selling off $(1 - \alpha^*) = 0.224$ of the firm, and using the net IPO proceeds to reinvest in the company. Note that the gross IPO proceeds equal P(.25,.776,0) = 14.15, and net proceeds are $P(.25,776,0)(1 - \rho) - F = 12.66$. When these proceeds are added to the initial capital $k = 12.91 < k^* = 25$, so it follows that $\beta V'(k + P(k, \alpha^*, 0)(1 - \rho) - F) > 1$, confirming the conclusion of Theorem 9 that it is better for the founder to reinvest all of the net proceeds from the IPO back into the firm rather than take any of this out in cash. Further, since the post-IPO public firm is still in the "liquidity constrained region" (the region where $k < \underline{k}$ where \underline{k} is the threshold defined in equation (7) of section 2), no dividends will be paid after the IPO.

Note that after the IPO, if the newly formed public company does not have access to debt markets and thus cannot borrow, it may want to raise the additional capital to reach the optimal steady state capital stock k^* as quickly as possible by doing an SEO. However referring to figure 7, even under the best case "Modigliani-Miller" scenario where the firm either has access to unlimited credit can do an SEO at zero cost, the gains to using additional external sources of finance are quite small once k = 12.91, compared to financing the remaining capital needs by retained earnings. Thus if SEO costs involve a commission of $\rho = 0.07$ and F = 0.5, in this particular example it would not be optimal for the post-IPO firm to turn around and do a follow-on SEO as well.

We are now in position to characterize the conditions under which it is optimal for the owner of a private firm to go public with an optimally structured IPO. At the risk of some notational confusion, we define the function $W_{ipo}(k)$ which unlike the function with the same name defined in equation 54 above,

Figure 16: Gains to going public, $W_{ipo}(k) - W(k)$, and $\alpha^*(k)$ when $(\rho, F) = (.07, .5)$



but now instead of the total *wealth* of the founder, we let $W_{ipo}(k)$ represent the *discounted utility* of the founder after an "optimal" IPO and after this resulting wealth is converted into an annuity, i.e.

$$W_{ipo}(k) = \frac{u((1-\beta)[\max_{\alpha,\omega} W(k,\alpha,\omega)])}{(1-\beta_p)}.$$
(56)

In fact, we can show that the owner is also better off from conducting an optimal IPO rather than continuing to run his company as a private firm: $W_{ipo}(k) > W(k)$ when k = 0.25.

The left hand panel of figure 16 plots the two value functions W(k) (the value function for the private firm owner without the option of going public, equation (46) in section 5) and $W_{ipo}(k)$ defined in equation (56) above. Due to our assumption that the fixed costs of the IPO must be paid up front, the IPO is not feasible until the capital stock exceeds $\underline{k} = .25$ since $f(\underline{k}) = F$. Let \overline{k} be the smallest value of k for which $W_{ipo}(\overline{k}) = W(\overline{k})$. Then we see from figure 16 that for $k \in (\underline{k}, \overline{k}) = (.25, 17.31)$ we have $W_{ipo}(k) > W(k)$ and this constitutes the "goldilocks region" where it is optimal for the owenr to go public. For $k > \overline{k}$ and sufficiently close to the optimal steady state capital stock $k^* = 25$, we have $W(k) > W_{ipo}(k)$, i.e. it is optimal for the owner to remain private in this region, since the costs of doing an IPO outweigh the benefits.

The right hand panel of figure 16 plots the optimal dilution value $\alpha^*(k)$. We see it increases monotonically from $\alpha^*(.25) = .77$ at the lower threshold $\underline{k} = .25$ for the "stopping region" where it is optimal to go public, and reaches $\alpha^*(k) = 1$ at k = 11.22. Thus, the financing value of doing an IPO quickly disappears as k increases and the owner's motive for going public switches to the desire for liquidity and eliminating what we previously terms the dividend smoothing inefficiency of private ownership. There is a higher threshold of capital, $\hat{k} > k^*$ not shown in figure 16 beyond which it is optimal for the firm to go public again, i.e. $W_{ipo}(k) > W(k)$ for $k > \hat{k} = 35.11$. Why would this be the case?

The reason is that when $k > \hat{k} > k^*$ the private owner is "overcapitalized" since his capital stock is above the optimal steady state value of $k^* = 25$. Referring back to figure 9, we see that due to the dividend smoothing inefficiency the private owner still invests at $k = \hat{k}$ but at a value that not enough to offset depreciation, so the capital stock will gradually shrink back to k^* . This implies that dividends will be slowly declining while the capital shrinks, leading the private owner to experience a temporary period of declining dividends and thus consumption. Because of our putty-clay assumption on the capital stock, the private owner cannot simply liquidate a portion of the capital stock to smooth out his consumption stream. Instead, it is optimal for the owner to take his company public so that it can be managed more efficiently and enable the owner to capture part of this efficiency gain (net of the cost of the IPO) and more efficiently smooth his consumption stream using financial markets (e.g. annuities). Refer back to figure 11 which illustrates the relatively small efficiency gains to be had from going public even by overcapitalized private firms.

To fully and rigorously define the optimal decision making of a private owner when he has the option of going public, we solve the Bellman equation given in equation (55) of section 5 for the 100% sell out case ($\alpha = 1$) but with $W_{ipo}(k)$ replaced by the value under an optimally done IPO givem in equation (56) above. We summarize this discussion in

Theorem 10 (the IPO decision as an optimal stopping problem) Assume neither the public or private firms have access to debt markets, but can raise external equity via a costly IPO with commission rate $\rho > 0$ and fixed costs F > 0. Let $\gamma(k)$ be the private owner's optimal stopping rule, where $\gamma(k) = 1$ denoting a decision to go public with an optimally chosen IPO and $\gamma(k) = 0$ denoting the decision to remain private. We have

$$\gamma(k) = \begin{cases} 0 & \text{if } k \leq \underline{k} \\ 1 & \text{if } k \in (\underline{k}, \overline{k}) \\ 0 & \text{if } k \in [\overline{k}, \hat{k}) \\ 1 & \text{if } k > \hat{k} \end{cases}$$
(57)

where $\underline{k} < \overline{k} < k^* < \hat{k}$.

We regard the possibility private firm might be overcapitalized (i.e. the case $k > \hat{k}$ in Theorem 10) to be a theoretical curiosum, since normal investment dynamics should never lead a private firm to become overcapitalized in the first place. It could possibly be relevant if an exogenous technological improvement enabled the firm to adopt a new more technologically efficient production function $f_1(k)$ in place of their existing production technology f(k) and the new production technology has an optimal steady state capital stock k_1^* that is less than the steady state capital k^* for production function f(k). In such situations it may be that overcapitalization is empirically relevant and cold be a motive for even a large private firm to go public, especially if there are large investment costs involved to convert the firm from production technology f to the new technology f_1 .



Figure 17: Gains to going public, $W_{ipo}(k) - W(k)$, and $\alpha^*(k)$ when $(\rho, F) = (.2, 2)$

As we noted from the work of Ritter [1987] in section 2.2 the cost of an IPO is much higher when the effect of underpricing of new issues is taken into account. Figure 17 shows the gains to going public when we set $(\rho, F) = (.2, 2)$, where the $\rho = .2$ value was motivated by low end of the estimates of Ritter [1987] on the cost of an IPO. We see that the increase in IPO costs dramatically shrinks the range of capital stocks over which it is optimal for the private owner to go public, from $[\underline{k}, \overline{k}] = [.25, 17.36]$ when $\rho = .07$ and F = .5 to $[\underline{k}, \overline{k}] = [2,9.94]$, or decreasing the width of this interval by more than 50%. We also see from the right hand panel of figure 17 that the share of the firm that the owner is willing to sell to outside investors is significantly reduced as well, from 15% when k = 4 and $(\rho, F) = (.07, .5)$ to 8% when $(\rho, F) = (.2, 2)$.

5.2 IPOs in the non-concave case

This section briefly describes a complication that can occur in the case of non-concave production functions f(k), such as the example we studied in section 3.1. A key issue is the potential for multiple fixed points to the condition defining SEO and IPO proceeds in equation (28) of section 3.3. Figure 18 illustrates the problem, for a case where there are a total of 4 IPO fixed points.

We argue that the solution to this problem is to *pick the largest fixed point*. In this case the largest fixed point results in gross IPO proceeds of $P(k, \alpha, \omega) = 234.03$ when k = 0, $\alpha = .35$ and $\omega = 0$. The reason why we choose this fixed point is 1) the owner strictly prefer larger IPO proceeds, and 2) we assume that via the IPO prospectus that the owner can credibly signal to outside investors the level of reinvestment he plans for the IPO proceeds, and this in effect acts as an "equilibrium selection mechansim". Thus, we argue that the private owner has the ability to select the IPO "equilibrium" which is best for him in cases where there may be multiple IPO fixed points to equation (28). After this small adjustment, our analysis of IPOs for firms with non-concave production functions proceeds in the same way as we did for a firm



Figure 18: Mulitple IPO fixed points, non-concave production function f(k)

with a concave production function.

We conclude this section by noting the IPOs are particularly effective and valuable for firms that have non-concave production functions. As we illustrated in figure 10, when a private firm has a non-concave production function there are large *growth traps* that can forever condemn the firm to convering to the small, less profitable steady state capital stock instead of the larger much more highly profitable steady state value of k^* . This is also true for a public firm, but the problems are more serious for private firms due to the dividend smoothing inefficiency: comparing the optimal investment functions in figure 10 (private firm) and figure 6 (public firm), we noted that the domain of attraction to the lower steady state capital stock $k^* = 25$ is much larger for the private firm compared to the public firm. Further as we noted in section 3.1, a public firm that has unlimited ability to borrow can nearly triple its equity value by borrowing enough to move from the low steady state $k^* = 25$ to the high steady state $k^* = 222.24$.

Thus even if neither the public or private firm had access to credit markets, as long as the private firm can go public, the owner can choose an optimal IPO ($\alpha^*(k), \omega^*(0)$) that can enable the owner to raise enough capital in the IPO and jump to the more profitable high capital stock steady state. Thus, IPOs are particularly valuable to private firms that have non-concave production functions and many "tech firms" may have production functions of this type. That is, it may be relatively easy and require relatively little investment to develop a software product and develop an initial, profitable customer base. But it may take substantial additional investment to market the product and "network" the software so that it achieves a dominant market share and huge profitability. Facebook may be an example of a tech firm with this sort of production function.

6 Extensions and Caveats

Not yet completed.

7 Conclusions

This paper introduced a simple theory of why and when firms go public. These are our main results:

Why go public? Because risk aversion and the desire of a private owner to consumption smooth causes him/her to distort their investment and financial policy, resulting in slower growth and less ambitious operation of the company. Though the behavior of the owner is *privately optimal* it is sub-optimal with respect to the criterion of maximization of market value of the firm. Provided the fixed and variable transaction costs of undertaking an IPO are not too high, the owner can gain by taking the firm public so it can be run by a risk-neutral manager in a way that maximizes the present value of dividends, and thus the equity valuation of the company.

When to go public? We showed that especially for firms with a non-concave production function (which we argued is relevant for firms that undergo multiple stages of growth as they evolve from a small scale firm with little capital to a large scale firm with a large capital stock that can exploit significant global returns to scale from expansion), the optimal time to go public is when the firm is at the "Goldilocks size" — not too small and not too big. Firms that are too small and which face borrowing constraints that prevent them from investing to quickly reach the efficient scale of operations will not experience large returns from going public, and will not be able to afford the significant fixed and proportional transactions costs of an IPO. Firms that have already acquired a large capital stock can afford to do an IPO but have already have nearly the level of capital they need to to achieve the efficient scale of operation, and so can easily use a combination of debt financing and retained earnings to acquire the remaining capital necessary to achieve their optimal scale.

Our theory is only a beginning. There are many ways to extend our simple model to make it more realistic such as adding uncertainty and learning as we discussed in section 6. Our eventual goal is to use this theory as a basis for *structural estimation* of the IPO decision using panel data on Indian firms over the period 1990 to 2005. Such an exercise can help us determine whether our theory really is capable of providing a good approximation to the behavior of actual firms, and explaining why some firms go public and others don't.

Our theory can be extended to incorporate the impact of various types of regulations and policies affecting IPOs, including regulations such as the Sarbanes-Oxley Act that was passed in the wake of

the Enron debacle and was designed to improve accounting, financial disclosures, company oversight and corporate responsibility. It remains an open question whether well intended regulations designed to prevent corporate abuses have had unintended effects that have contributed to the decline in IPOs. As Ritter argues, "It is possible that, by making it easier to raise money privately, creating some liquidity without being public, restricting the information that stockholders have access to, restricting the ability of public market shareholders to constrain managers after investors contribute capital, and driving out independent research, the net effect of the JOBS Act might be to reduce the flow of capital into young high-technology companies or the number of IPOs of small emerging growth companies."

However our theory suggests it is not just regulatory burdens that could contribute to the trend: we find that the costs of undertaking IPOs, with the large 7% underwriting commission and associated fixed costs, can also be a significant deterrent of using IPOs to finance growth relative to debt financing or retained earnings. But most importantly it is the owner's *expectations* and beliefs about the potential gains to significant capital investments that is a decisive driver of the decision to do an IPO. If competition in the product market is tougher making it harder for newly ascending companies to break the "glass ceiling" of competition from larger well established rivals, then even drastic reductions or even outright elimination of regulatory burdens and reductions in the investment banking costs of doing IPOs (perhaps via policies that encourage freer entry into the underwriting sector) may not have a significant effect on the number of IPOs, especially with the expansion of "private capital" and "venture capital" as alternative potentially more efficient alternative to IPOs. As Decker *et. al.* note, "Theo open question is why firms with high realizations of productivity, especially those in the High Tech sector, do not experience the same high growth as before."

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